REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

AD-A240 773

2. REPORT DATE: August 1991

3. REPORT TYPE AND DATES COVERED: Professional paper

4. TITLE AND SUBTITLE: ALGEBRAIC AND PROBABILISTIC BASES FOR FUZZY SETS AND THE DEVELOPMENT OF FUZZY CONDITIONING

5. FUNDING NUMBERS

PR: ZE90	PR: CD32
PE: 0602936N	PE: 0305108K
WU: DN300036	WU: DN488828

6. AUTHOR(S)

I. R. Goodman

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Ocean Systems Center
San Diego, CA 92152-5000

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office Chief of Naval Research
Independent Exploratory Development Programs
(IED) OCNR-20T
Arlington, VA 22217

Office of the Secretary of Defense
Research and Engineering
Washington, DC 20363

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

This paper first develops an extension of the Negoita-Ralescu Representation Theorem for fuzzy sets in terms of flou sets relative to operators and partitionings. It then reviews in some detail both the random set/random variable basis for fuzzy sets, as well as the foundation of conditional event algebras. Both of these areas are tied together, first in the form of conditional event indicator functions, and then through the development of conditioning fuzzy sets. Specifically, it is shown that the structure of conditional event algebra as proposed here drives the structure for fuzzy conditioning, resulting in conditional fuzzy sets being necessarily of a simple form relative to their membership functions to a given marginal. It is seen that with this approach, a full calculus of operations, extending that of ordinary conditional events, is obtained.

Published in the book *Conditional Logic in Expert Systems*, North-Holland Publishers, 1991.

14. SUBJECT TERMS

data fusion
conditional events
conditional event algebra

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAME AS REPORT

UNCLASSIFIED

21a. NAME OF RESPONSIBLE INDIVIDUAL	21b. TELEPHONE (Include Area Code)	21c OFFICE SYMBOL
I. R. Goodman	(619) 553–4014	Code 421

NSN 7540-01-280-5500

Standard form 298

UNCLASSIFIED

Conditional Logic in Expert Systems
I.R. Goodman, M.M. Gupta, H.T. Nguyen and G.S. Rogers (editors)
Elsevier Science Publishers B.V. (North-Holland), 1991

ALGEBRAIC AND PROBABILISTIC BASES FOR FUZZY SETS AND THE DEVELOPMENT OF FUZZY CONDITIONING

I.R. Goodman

Code 421, Command & Control Department
Naval Ocean Systems Center
San Diego, CA 92152-5000

Abstract. This paper first develops an extension of the Negoita-Ralescu Representation Theorem for fuzzy sets in terms of flou sets relative to operators and partitionings. It then reviews in some detail both the random set/random variable basis for fuzzy sets, as well as the foundation of conditional event algebras. Both of these areas are tied together, first in the form of conditional event indicator functions, and then through the development of conditioning fuzzy sets. Specificially, it is shown that the structure of conditional event algebra as proposed here drives the structure for fuzzy conditioning, resulting in conditional fuzzy sets being necessarily of a simple form relative to their membership functions to a given marginal. It is seen that with this approach, a full calculus of operations, extending that of ordinary conditional events, is obtained.

Keywords. Fuzzy sets, membership functions, flou sets, conditional fuzzy sets, random sets, partitionings, conditional event algebra.

1. Introduction.

Even after twenty-five years following Zadeh's introduction of fuzzy sets (1965), controversy still persists in this arena of uncertainty modeling: 1. Should one choose a fuzzy set or probability approach to a particular problem at hand? 2. Can objective criteria be set up for comparing and contrasting fuzzy sets and probability? 3. What, exactly, are the relations between the two approaches and can they be reconciled with each other? 4. Can an analogue of conditioning in probability be established for fuzzy sets, especially in light of the newly-developed area of conditional event algebra (Goodman & Nguyen, (1988), Goodman, Nguyen, Walker (1991))?

The first question still remains an open issue to this day. An approach to answering the second one has been done through the use of game theory, as proposed by Lindley (1982) and reconsidered by Goodman, Nguyen & Rogers (1991). As for question three, previously Goodman (1981), Hohle (1982), and Goodman & Nguyen (1985), among others, initiated work on relating directly fuzzy sets and probability through random set theory. In another direction, Negoita & Ralescu have considered the relationship between fuzzy sets and certain collections of nested ordinary sets (or "flou" sets) (1975), while Gaines has considered both fuzzy sets and probability logic from a common algebraic framework (1978). (See Goodman & Nguyen (1985, Chapter 7) for a more thorough history of attempts at connecting fuzzy sets with probability.) The last question has been addressed by a number of individuals. E.g., Mattila (1986), Sembi & Mamdani (1979), and Yager (1983) consider extensions and modifications of ordinary material implication, while Zadeh (1978), Nguyen (1978), Hisdal (1978), Bouchon (1987), and Goodman & Stein (1989) approached fuzzy conditioning with at least some concept of conditional probability relative to ordinary sets in mind.

A common theme underlies the above issues and their responses: there is a real need to, once and for all, establish a unifying approach to fuzzy sets, their algebraic or syntactic bases, and their internal and external relations to probability Recently, conditioning in probability has been re-examined and it has been demonstrated that a firm algebraic basis -- in addition to the usual numerically-oriented approach -- can be derived for conditioning. (See Schay (1968), Adams (1975), Calabrese (1987), and Goodman, Nguyen, & Walker (1991), as well as the work of Dubois & Prade (1990).) Thus,it would also be desirable to be able to extend the above work to fuzzy sets based on firm logical considerations.

The purpose of this paper is twofold: First, to develop a sound algebraic basis for fuzzy sets, based upon the fundamental work of Negoita & Ralescu (1975). This will serve as a lead-in to the probability basis for fuzzy sets. In short, flou sets -- and a new alternative, but equivalent, representation in the form of ordered partitionings -- are proposed as the natural candidates for the syntactic foundations of fuzzy sets, underlying the semantic evaluations: fuzzy set membership functions. However, the scope here is a limited one and the very generalized set theory encompassing fuzzy sets in the form of categories and pseudotopoi will not be treated here. (See, e.g., Barr (1989), Eytan (1981), Pitts (1982), Goguen (1974), and Stout (1984).)

In addition, extensions of the Stone Representation Theorem to fuzzy sets as, e.g., treated in Glas (1984) and Belluce (1986) will not be considered. The second goal of this paper is to be able to apply the basic algebraic and probabilistic foundations for fuzzy sets to the development of conditioning and related concepts.

This paper consists of eight additional sections. In section 2 the basic spaces are considred: partitioning, flou, and membership function spaces and their bijections. In section 3 a standard procedure is reviewed for inducing isomorphisms from bijections relative to the base spaces. Section 4 develops operations isomorphic to fuzzy set membership operations, including cartesian products, sums, intersections, unions, complements, functional and inverse functional transforms, among others. A similar development for partitioning sets is given in section 5. Section 6 reviews briefly conditioning of ordinary sets and establishes a connection with three-valued fuzzy set membership functions as a special case of finite-valued membership functions. In section 7, logical models for fuzzy sets are characterized. In turn, external probabilities of fuzzy set membership functions are determined. These are especially useful as a rationale for single figures-of-merit for fuzzy sets -- analogous to the moments of cdf's. In a direction opposite to section 7, the underlying probability basis for fuzzy sets is summarized in section 8. The focus here is the uniform randomization of flou sets and partitioning sets, as well as their isomorphic relations to the class of membership functions. (A third connection between probability and fuzzy sets is given briefly at the end of sect. 4 via cdf's as formal fuzzy set membership functions.) Finally, in section 9 conditional fuzzy sets are defined, based upon random set considerations as developed in the previous sections. A full calculus of operations and relations is derived, extending all of the previous results obtained for ordinary conditional events to fuzzy sets.

2. Fundamental Spaces and Bijective Mappings.

Throughout the remaining paper denote the unit interval $[0, 1] = \{t : 0 \le t \le 1\}$ by u. Also, let SET denote the collection of all well-defined sets and consider the operators Part, Flou, Mem:SET $\to$ SET and mappings on SET, ϕ, ψ, where for all $X \in$ SET, $\phi(X) : \text{Flou}(X) \to \text{Mem}(X)$ and $\psi(X) : \text{Part}(X) \to \text{Flou}(X)$.

$\text{Part}(X) \overset{d}{=}$ set of all *ordered disjoint nonvacuous exhaustive partitionings* q of X, where typically (2.1)

$$q = (q_t)_{t\in J_q}, \; \emptyset \ne J_q \subseteq u \; ; \; q_t \in \mathcal{P}(X); \; q_s \cap q_t = \emptyset, \; s \ne t; \; \bigcup_{t\in J_q} q_t = X. \tag{2.2}$$

$\text{Flou}(X) \overset{d}{=}$ set of all *flou sets* (see originally Gentilhomme (1968)) a of X, where typically (2.3)

$$a = (a_t)_{t\in u}, \; X = a_0 \supseteq a_s \supseteq a_t \supseteq a_1 \supseteq \emptyset; \; \bigcap_{t\in J} a_t = a_{\sup(J)}, \text{ all } J \subseteq u; \; 0 \le s \le t \le 1 \tag{2.4}$$

arbitrary real. The right hand side relation is continuity from above.

$$\text{Mem}(X) \overset{d}{=} \text{set of all } \textit{fuzzy set membership functions } f \text{ of } X$$
$$= u^X = \{f : f : X \to u\}, \tag{2.5}$$

including all ordinary set indicator functions $g : X \to \{0, 1\}$

$\phi(X) : \text{Flou}(X) \to \text{Mem}(X)$, the *fundamental membership mapping* is defined for any

$a \in \mathrm{Flou}(X)$, $\phi(X)(a) : X \to u$, where for all $x \in X$,

$$\phi(X)(a)(x) \overset{d}{=} \sup\{t : t \in u \ \& \ x \in a_t\}. \tag{2.6}$$

$\psi(X) : \mathrm{Part}(X) \to \mathrm{Flou}(X)$ is the *fundamental fuzzy set forming mapping*, where for any $q \in \mathrm{Part}(X)$ and any $t \in u$,

$$(\psi(X)(q))_t \overset{d}{=} \cup\{q_s : s \in J_q \ \& \ t \le s \le 1\}. \tag{2.7}$$

All of this leads to

Theorem 2.1. For each $X \in \mathrm{SET}$, $\phi(X)$ is a bijection, with inverse $\phi(X)^{-1}$: $\mathrm{Mem}(X) \to \mathrm{Flou}(X)$ given for any $f \in \mathrm{Mem}(X)$ as $\phi(X)^{-1}(f) \in \mathrm{Flou}(X)$, where for all $t \in u$,

$$(\phi(X)^{-1}(f))_t \overset{d}{=} f^{-1}[t, 1] = \{x : x \in X \ \& \ 1 \ge f(x) \tag{2.8}$$

the t^{th}*-level* (or *cut*) *set* of f. Note also that for all $x \in X$, the supremum in eq. (2.6) is always achieved, so that

$$x \in a_{\phi(X)(a)(x)}, \quad \text{all } x \in X. \tag{2.9}$$

Proof: Though Negoita & Ralescu (1975) have developed a representation theorem with a slightly different form, for purposes of completeness, a full proof for the present version will be presented here.

Obviously, $\phi(X)$ is well-defined. For any $f \in \mathrm{Mem}(X)$, define $\phi(X)^{-1}(f)$ as in (2.8). Clearly, from the basic properties of inverse functions, $\phi(X)^{-1}(f)$ satisfies property left hand side of (2.4). For the right hand side of (2.4) let $J \subseteq u$ arbitrary (nonvacuous). Then, for any $x \in X$, $x \in \cap\{f^{-1}[t, 1] : t \in J\}$ iff $f(x) \ge J$ iff $f(x) \ge \sup(J)$ iff $x \in f^{-1}[\sup(J), 1]$. Thus, (2.4) is satisfied and $\phi(X)^{-1}(f) \in \mathrm{Flou}(X)$. In turn, for any $x \in X$, $\phi(X)(\phi(X)^{-1}(f))(x) = \sup\{t : t \in u \ \& \ x \in f^{-1}[t, 1]\}$ $= \sup\{t : t \in u \ \& \ t \le f(x)\} = f(x)$, implying that $\phi(X)$ is surjective with $\phi(X)^{-1}$ being a candidate for its inverse. Next, let $a \in \mathrm{Flou}(X)$ arbitrary and for any $t \in u$,

$$\begin{aligned}(\phi(X)^{-1}(\phi(X)(a)))_t &= \phi(X)(a)^{-1}[t, 1] = \{x : x \in X \ \& \ \phi(X)(a)(x) \ge t\} \\ &= \{x : x \in X \ \& \ \sup\{s : s \in u \ \& \ x \in a_s\} \ge t\}. \end{aligned} \tag{2.10}$$

Now, if $x \in a_t$, then clearly $\sup\{s : s \in u \ \& \ x \in a_s\} \ge t$. Conversely, if the $\sup \ge t$, then letting $J_x \overset{d}{=} \{s : s \in u \ \& \ x \in a_s\}$, by RHS (2.4) property,

$x \in \cap\{a_s : s \in J_x\} = a_{\sup(J_x)}$ with $\sup(J_x) \geq t$, whence $x \in a_{\sup(J_x)} \subseteq a_t$.

Thus, for all $t \in u$, $x \in a_t$ iff $\sup\{s : s \in u \ \& \ x \in a_s\} \geq t$; all $x \in X$. (2.11)

Combining (2.10) and (2.11) shows

$$(\phi(X)^{-1}(\phi(X)(a)))_t = \{x : x \in a_t\} = a_t, \tag{2.12}$$

verifying that for all $a \in \text{Flou}(X)$,

$$\phi(X)^{-1}(\phi(X)(a)) = a. \tag{2.13}$$

It is readily seen that (2.13) is sufficient to show that $\phi(X)$ is injective. Since $\phi(X)$ was also shown to be surjective, the above shows that it is bijective. Finally, (2.13) also shows (2.9) directly. ■

Theorem 2.2. $\psi(X)$ is a bijection with inverse $\psi(X)^{-1} : \text{Flou}(X) \to \text{Part}(X)$, given for any $a \in \text{Flou}(X)$ as $\psi(X)^{-1}(a) \in \text{Part}(X)$, with index set

$$J_{\psi(X)^{-1}(a)} = \{t : t \in u \ \& \ a_t - a_{t^+} \neq \emptyset\}, \tag{2.14}$$

where

$$a_{t^+} = \cup\{a_s : t < s \leq 1\}, \ t \in u, \tag{2.15}$$

and where for all $t \in J_{\psi(X)^{-1}(a)}$, i.e., $a_t - a_{t^+} \neq \emptyset$,

$$(\psi(X)^{-1}(a))_t \overset{d}{=} a_t - a_{t^+}, \tag{2.16}$$

with the convention that

$$a_{1^+} = \underset{\emptyset}{\cup}\{a_s\} = \emptyset. \tag{2.17}$$

Proof: First, note that for any $q \in \text{Part}(X)$, and hence $\psi(X)(q) \in \text{Flou}(X)$: For all $0 \leq s \leq t \leq 1$,

$$(\psi(X)(q))_0 = \cup\{q_s : s \in J_q\} = X \ ; \ (\psi(X)(q))_s = \cup\{q_r : r \in J_q, s \leq r\}$$
$$\supseteq \cup\{q_r : r \in J_q, t \leq r\} = (\psi(X)(q))_t, \tag{2.18}$$

verifying the left hand side of (2.4). For any $K \subseteq u$, let $x \in (\psi(X)(q))_{\sup(K)}$. Thus, there exists $s \in J_q$ with $s \geq \sup(K)$ such that $x \in q_s$. Hence, for each $t \in K$, there exists $s \in J_q$ with $s > \sup(K)$ and $x \in q_s$. Hence,

$$x \in (\psi(X)(q))_{\sup(K)} \subseteq \bigcap_{t \in K} \bigcup_{t \leq s \leq 1} q_s = \bigcap_{t \in K} (\psi(X)(q))_t \tag{2.19}$$

Conversely, let $x \in \bigcap_{t \in K} (\psi(X)(q))_t$. Since q is a partitioning of X, there is a unique $t_o \in J_q$ such that $x \in q_{t_o}$. Thus, $x \in \bigcap_{t \in K} (\psi(X)(q))_t$ becomes: for all $t \in K$, $x \in (\psi(X)(q))_t$, so that for each $t \in K$, there is an $s \in J_q$ with $t \leq s$, $x \in q_s = q_{t_o}$, noting that $t_o \geq K$. Hence

$$x \in q_{t_o} \subseteq \bigcup_{s\in[\sup(K),1]\cap J_q} (q_s) = (\psi(X)(q))_{\sup(K)} \tag{2.20}$$

Combining (2.19) and (2.20) shows the right hand side of (2.4) holding. Hence (2.4) completely holds and $\psi(X)(q) \in \mathrm{Flou}(X)$. Hence, $\psi(X) : \mathrm{Part}(X) \to \mathrm{Flou}(X)$ is a well-defined mapping.

Next, consider the mapping $\phi(X)\circ\psi(X) : \mathrm{Part}(X) \to \mathrm{Mem}(X)$ which is also well-defined since ϕ and ψ are. For any $f \in \mathrm{Mem}(X)$, consider the partitioning

$$q(f) = ((q(f))_s)_{s\in J_f} \; ; \; J_f \overset{d}{=} \mathrm{range}(f) = \{f(x) : x \in X\} \; ; \; (q(f))_s \overset{d}{=} f^{-1}(s), \tag{2.21}$$

for all $s \in J_f$. Then, for all $x \in X$, using (2.21),

$$\phi(X)(\psi(X)(q(f)))(x) = \sup\{t : t \in u \;\&\; x \in \bigcup_{(s\in J_f, t\leq s)} f^{-1}(s)\}$$

$$= \sup\{t : t \in u \;\&\; x \in f^{-1}[t, 1]\} = f(x), \tag{2.22}$$

showing $\phi(X)\circ\psi(X)$ is surjective with

$$(\phi(X)\circ\psi(X))(q(f)) = f, \text{ all } f \in \mathrm{Mem}(X). \tag{2.23}$$

next, for each $q \in \mathrm{Part}(X)$, define $f_q \in \mathrm{Mem}(X)$ by, for all $x \in X$,

$$f_q(x) \overset{d}{=} s, \text{ for that unique } s \in J_q \text{ for which } x \in q_s \tag{2.24}$$

Clearly, (2.24) is equivalent to the relation

$$f_q^{-1}(s) = q_s \,, \text{ all } s \in J_q. \tag{2.25}$$

Note, using the notation of (2.21), $J_{f_q} = \mathrm{range}(f_q) = J_q$, and since for all $s \in J_{f_q}$, (2.25) shows $(q(f_q))_s = f_q^{-1}(s) = q_s$, then one has

$$a(f_q) = q. \tag{2.26}$$

Finally, replacing f by f_q in (2.23), using (2.26), shows that

$$(\phi(X)\circ\psi(X))(q) = f_q. \tag{2.27}$$

In turn, (2.27) shows that $\phi(X)\circ\psi(X)$ is also injective. Hence, by the previously established property of being surjective, $\phi(X)\circ\psi(X)$ is bijective.

Next, (2.23) in conjunction with the bijectivity of $\phi(X)\circ\psi(X)$ shows

$$\psi(X)^{-1}\circ\phi(X)^{-1}(f) = (\phi(X)\circ\psi(X))^{-1}(f) = q(f). \tag{2.28}$$

Then, letting $a \in \mathrm{Flou}(X)$ arbitrary and choosing $f = \phi(X)(a)$ in (2.28), since by Theorem 2.1, $\phi(X)^{-1}(\phi(X)(a)) = a$, one obtains

$$\psi(X)^{-1}(a) = q(\phi(X)(a)), \tag{2.29}$$

where by (2.21)

$$J_{q(\phi(X)(a))} = \text{range}(\phi(X)(a)) = \{\sup\{t : t \in u \ \& \ x \in a_t\} : x \in X\}, \tag{2.30}$$

and for each $t \in J_{q(\phi(X)(a))}$, by (2.24),

$$(q(\phi(X)(a)))_t = (\phi(X)(a))^{-1}(t). \tag{2.31}$$

But, Theorem 2.1 shows

$$a_t = (\phi(X)(a))^{-1}[t, 1] \tag{2.32}$$

and

$$\begin{aligned} a_{t^+} &= \bigcup_{t<s\le 1} a_s = \bigcup_{t<s\le 1} (\phi(X)(a))^{-1}[s,1] \\ &= (\phi(X)(a))^{-1}\Big(\bigcup_{t<s\le 1} [s, 1]\Big) = (\phi(X)(a))^{-1}(t,1]. \end{aligned} \tag{2.33}$$

Combining (2.29)-(2.33), shows for all $t \in J_{\psi(X)^{-1}(a)}$,

$$\begin{aligned} (\psi(X)^{-1}(a))_t &= (\phi(X)(a))^{-1}(t) = (\phi(X)(a))^{-1}[t, 1] - (\phi(X)(a))^{-1}(t, 1] \\ &= a_t - a_{t^+}, \text{ matching eq. (2.16).} \end{aligned} \tag{2.34}$$

Finally, by (2.29) and (2.31),

$$a_t - a_{t^+} = (\phi(X)(a))^{-1}(t) \neq \emptyset \ \text{ iff } \ t \in \text{range}(\phi(X)(a)) = J_{q(\phi(X)(a))}. \tag{2.35}$$

Eq. (2.35) shows (2.14). ∎

The proof technique of Theorem 2.2 leads immediately to

Corollary 2.1. $\phi(X)\circ\psi(X) : \text{Part}(X) \to \text{Mem}(X)$ is a bijection, where $\phi(X)\circ\psi(X)$ can be expressed as in eqs. (2.27) and (2.24), with inverse $(\phi(X)\circ\psi(X))^{-1}$: $\text{Mem}(X) \to \text{Part}(X)$, which can be expressed, using (2.23) as

$$(\phi(X)\circ\psi(X))^{-1}(f) = \psi(X)^{-1}\circ\phi(X)^{-1}(f) = q(f). \tag{2.36}$$

∎

Summarizing, the following diagram of bijections holds:

$$\begin{tikzcd} & \text{Flou}(X) \arrow[dr, "\phi(X)"] & \\ \text{Part}(X) \arrow[ur, "\psi(X)"] \arrow[rr, "\phi(X)\circ\psi(X)"'] & & \text{Mem}(X) \end{tikzcd}$$

Figure 2.1. Summary of bijections for Mem(X), Flou(X), Part(X).

The basic relationships are, omitting the (X) notation for ϕ, ψ, for all $q = (q_t)_{t\in J_q} \in \text{Part}(X)$, $a = (a_t)_{t\in u} \in \text{Flou}(X)$, $f \in \text{Mem}(X)$, and all $t \in u$, $x \in X$:

$$\left.\begin{aligned}&\phi(a)(x) = \sup\{t : t \in u \ \&\ x \in a_t\};\ (\phi^{-1}(f))_t = f^{-1}[t,1];\ (\psi(q))_t = \cup\{q_s : s \in J_q, t \le s \le 1\};\\ &J_{\psi^{-1}(a)} = \{t : t \in u \ \&\ a_t - a_{t^+} \neq \emptyset\};\ (\psi^{-1}(a))_t = a_t - a_{t^+},\ \text{all } t \in J_{\psi^{-1}(a)};\end{aligned}\right\}$$

$$\left.\begin{aligned}&(\phi_\circ\psi)(q)(x) = f_q(x) = s\ \ (\text{for that unique } s \in J_q, \text{ where } x \in q_s);\ J_{(\phi_\circ\psi)^{-1}(f)} =\\ &\text{range}(f);\ \ ((\phi_\circ\psi)^{-1}(f))_s = f^{-1}(s),\ \text{all } s \in \text{range}(f).\end{aligned}\right\} \qquad (2.37)$$

3. Isomorphisms among Operations over the Fundamental Spaces: Introduction.

First note the following general constructive procedure:

Let $(X, *)$ be a given space with operation $*$ over X which could be n-ary as $* : X^n \to X$ and let Y be any other (nonvacuous) set such that $\tau : X \to Y$ is a bijection. Then, define (n-ary) operation $\tau(*) : Y^n \to Y$ by

$$\tau(*)(y_1,\dots,y_n) \overset{d}{=} \tau(*(\tau^{-1}(y_1),\dots,\tau^{-1}(y_n))),\ \text{all } y_1,\dots,y_n \in Y, \qquad (3.1)$$

i.e.,
$$\tau(*) \overset{d}{=} \tau_\circ *_\circ \tau^{-1}\ \ (\text{n-ary})\ ; \qquad (3.2)$$

so that τ and $*$ commute through $\tau(*)$:

$$\tau(*(x_1,\dots,x_n)) = \tau(*)(\tau(x_1),\dots,\tau(x_n)),\ \text{all } x_1,\dots,x_n \in X, \qquad (3.3)$$

i.e., $(X, *)$ and $(Y, \tau(*))$ are *isomorphic through* τ. (A similar construction holds when τ^{-1} is replaced by, say, $\eta : Y \to X$ throughout eqs. (3.1)-(3.3).) Call $(Y, \tau(*))$ *the space induced isomorphically by bijection* τ.

We will apply the above procedure several times throughout the paper to determine the natural isomorphic counterparts among operators defined over Part(X), Flou(X), and Mem(X), *based on the traditional Zadeh and Zadeh-extended operators and relations with respect to Mem(X).* (See, e.g., the standard text by Dubois & Prade (1980) for background on these operators.) Specifically, the operators and relations to be considered here are: 1, cartesian products and their specialization to intersections; 2, cartesian sums and their specialization to unions; 3, subset relations; 4, complement operator; 5, attribute tranforms/functional extension principle; 6, inverse attribute transforms; 7, modifiers -- intensifiers and extensifiers. Conditioning, an important eighth type of operator will be considered separately in later sections, especially sections 6 and 10.

First, a brief note on the notation: Unless otherwise specified, $X, Y, Z, X_1, X_2, \dots, X_n, Y_1, \dots, Y_n \in$ SET arbitrary but fixed for any arbitrary but fixed positive integer n.

$T: X \to Y$ is any mapping and $T^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$ is its inverse mapping, where $\mathcal{P}(\;)$ denotes the power class of $(\;)$ or the class of all (ordinary) subsets of $(\;)$. $T_n : \overset{n}{\underset{j=1}{\times}} X_j \to Y$ is arbitrary as is $H: u \to u$ (recalling that u denotes the unit interval). Also, choose any continuous n-copula, i.e., cdf of an n by 1 r.v. representing the joint behavior of n one-dimensional marginal r.v.'s which are distributed uniformly over u. Thus, cop is the cdf for $\underline{\mathcal{U}} = (\mathcal{U}_1, ..., \mathcal{U}_n)$, where $\mathcal{U}_j : \Lambda \to u$, $j = 1, ..., n$, relative to some fixed probability space $(\Lambda, \mathcal{A}, p)$. Dually, denote the DeMorgan transform $1\text{-cop}(1 - (\;),...,1 - (\;))$ (n-ary operation) by cocop (cocopula). (See Schweizer & Sklar (1983) for general background.) In particular, note Zadeh's original copula min, as well as prod and minsum(only, 2-copulas) $\text{minsum}(s, t) \overset{d}{=} \max(s + t - 1, 0)$, all $s, t \in u$, as well as a wide variety of other examples as given in Goodman & Nguyen (1985, sect. 2.3.6). Three important examples of cocopulas are Zadeh's original max and probsum, and maxsum, where $\text{probsum}(s, t) \overset{d}{=} 1 - ((1 - s)\cdot(1 - t))$ and $\text{maxsum}(s, t) \overset{d}{=} \min(s + t - 1, 0)$ (the latter being only a 2-copula). (Again, see references above for further details.)

Also, let $f^{(1)}, f^{(2)}, f \in \text{Mem}(X)$, $g \in \text{Mem}(Y)$, and $f_j \in \text{Mem}(X_j)$, $j = 1, ..., n$ all arbitrary fixed. Use the multivariable notation

$\underline{f} \overset{d}{=} (f_1,...,f_n)$ (n arguments) ; $\underline{X} \overset{d}{=} (X_1,...,X_n)$; $\times\underline{X} \overset{d}{=} \overset{n}{\underset{j=1}{\times}} X_j$; $\underline{x} \overset{d}{=} (x_1,...,x_n) \in \times\underline{X}$,

i.e., $x_j \in X_j$, $j = 1, ..., n$; $\underline{f}(\underline{x}) \overset{d}{=} (f_1(x_1),...,f_n(x_n))$; for any $\underline{t} \overset{d}{=} (t_1,...,t_n) \in u^n$, $\text{cop}(\underline{t}) \overset{d}{=} \text{cop}(t_1,...,t_n)$. When $X = X_1 = .. = X_n$, $\underline{f}(x) \overset{d}{=} (f_1(x),...,f_n(x))$, $x \in X$.

The seven types of Zadeh -- and related -- fuzzy set operations and relations defined through the membership functions to be considered here are in summary:

(1) cartesian product of $\underline{f}$ wrt cop $\overset{d}{=} \times_{\text{cop}}(\underline{f}) \in \text{Mem}(\times\underline{X})$, (3.4)

$$\times_{\text{cop}}(\underline{f}) \overset{d}{=} \text{cop}(\underline{f}(\underline{x}))\,, \text{ all } \underline{x} \in \underline{X}. \tag{3.5}$$

In particular, for $X = X_1 = .. = X_n$,

intersection of $\underline{f}$ wrt cop $\overset{d}{=} \cap_{\text{cop}}(\underline{f}) \in \text{Mem}(X)$, (3.6)

$$\cap_{\text{cop}}(\underline{f})(x) \overset{d}{=} \text{cop}(\underline{f}(x)), \text{ all } x \in X. \tag{3.7}$$

(2) cartesian sum of $\underline{f}$ wrt cocop $\overset{d}{=} \dagger_{cocop}(\underline{f}) \in \mathrm{Mem}(\times\underline{X})$ (3.8)

$$\dagger_{cocop}(\underline{f})(\underline{x}) \overset{d}{=} \mathrm{cocop}(\underline{f}(\underline{x})), \text{ all } \underline{x} \in \underline{X}. \qquad (3.9)$$

In particular, for $X = X_1 = .. = X_n$,

$$\text{union of } \underline{f} \text{ wrt cocop} \overset{d}{=} \cup_{cocop}(\underline{f}) \in \mathrm{Mem}(X), \qquad (3.10)$$

$$\cup_{cocop}(\underline{f})(x) \overset{d}{=} \mathrm{cocop}(\underline{f}(x)), \text{ all } x \in X. \qquad (3.11)$$

(3) $f^{(1)}$ is in subset relation to $f^{(2)}$ iff, by def., $f^{(1)} \le f^{(2)}$ over X. (3.12)

(4) complement of $f \overset{d}{=} f' \overset{d}{=} 1 - f \in \mathrm{Mem}(X)$ (3.13)

(5) T-attribute transform of $f \overset{d}{=} T(f) \in \mathrm{Mem}(Y)$, (3.14)

$$T(f)(y) \overset{d}{=} \sup(f(T^{-1}(y))) = \sup_{x \in T^{-1}(y)} f(x) = \sup_{T(x)=y} f(x), \text{ all } y \in Y. \qquad (3.15)$$

In particular, for $X = \times\underline{X}$,

$$\text{T-attribute transform of } \underline{f} \text{ wrt cop} \overset{d}{=} T_{cop}(\underline{f}) \in \mathrm{Mem}(Y), \qquad (3.16)$$

$$T_{cop}(\underline{f})(y) \overset{d}{=} \sup(\times_{cop}(\underline{f})(T^{-1}(y))) = \sup_{\underline{x} \in T^{-1}(y)} (\times_{cop}(\underline{f})(\underline{x})). \qquad (3.17)$$

(6) T^{-1}-attribute transform of $g \overset{d}{=} T^{-1}(g) \in \mathrm{Mem}(X)$, (3.18)

$$T^{-1}(g) \overset{d}{=} g \circ T, \text{ i.e., } T^{-1}(g)(x) = g(T(X))), \text{ all } x \in X. \qquad (3.19)$$

(7) H-modifier of $f \overset{d}{=} H \circ f$, i.e., $(H \circ f)(x) = H(f(x))$, all $x \in X$. (3.20)

Note that though (6) and (7) look similar in form, (6) is the composition of the membership function on another (T), while (7) is the composition of a function (H, necessarily over u) on the membership function.

The next section constructs the isomorphic counterparts of the above over Flou(X).

4. Construction of Operations over Flou Spaces Isomorphic to Those over Fuzzy Set Membership Function Spaces.

Negoita & Ralescu (1975) and Ralescu (1979) were among the first to develop a full isomorphism between fuzzy set membership functions over a set endowed with Zadeh's original operations min for intersection or cartesian product and max for union or cartesian sum and flou (as nested collections of) sets with component-wise intersections and unions -- but not complements nor other operations. (This work extended the earlier work of Gentilhomme (1968) who introduced finite collections of nested sets as "flou" sets to explain multiple logic concepts through the use of ordinary sets, indeed without referring at all to Zadeh's still earlier pioneering effort (1965).) Radecki (1977) also considered independently a similar situation, emphasizing the level set forms of the nested sets relative to given membership functions.

In this section all of the above work is extended to include the seven types of operations and relations introduced in section 3. The resulting isomorphism from the procedure of section 3 applied to Theorem 2.1 show why it is natural to employ Flou(X) as the algebraic basis for fuzzy sets. In addition to the notation introduced in the previous section, denote $\underline{a} \overset{d}{=} (a^{(1)},...,a^{(n)}) \in \text{Flou}(\underline{X}) \overset{d}{=} (\text{Flou}(X_1),...,\text{Flou}(X_n))$, when $a^{(j)} \in \text{Flou}(X_j)$, $j = 1,...,n$ arbitrary. Similarly, denote $\underline{b} \overset{d}{=} (b^{(1)},...,b^{(n)}) \in \text{Flou}(Y)$, when $b^{(j)} \in \text{Flou}(Y_j)$, $j = 1,...,n$. Also $a \in \text{Flou}(X)$ and $b \in \text{Flou}(Y)$ are typical elements; $\phi(\underline{a}) \overset{d}{=} (\phi(a^{(1)}),...,\phi(a^{(n)}))$ (n arguments); for any $\underline{t} \in u^n$, $\times a_{\underline{s}} \overset{d}{=} a^{(1)}_{s_1} \times..\times a^{(n)}_{s_n}$, etc. For clarity, bold face is used on some operations:

(1) cartesian product of $\underline{a}$ wrt cop $\overset{d}{=} \times_{cop}(\underline{a}) \in \text{Flou}(\times\underline{X})$, (4.1)

$$(\times_{cop}(\underline{a}))_t \overset{d}{=} (\phi^{-1}(\times_{cop}(\phi(\underline{a}))))_t = (\times_{cop}(\phi(\underline{a})))^{-1}[t, 1]$$

$$= \bigcup_{\substack{\text{over all } \underline{s} \in u^n, \\ cop(\underline{s})=t}} (\times a_{\underline{s}}), \tag{4.2}$$

for all $t \in u$. Intersection becomes for $X = X_1 =..= X_n$,

$$\cap_{cop}(\underline{a}) \in \text{Flou}(X); (\cap_{cop}(\underline{a})))_t = \bigcup_{\substack{\text{over all } \underline{s} \in u^n, \\ cop(\underline{s})=t}} (\cap a_{\underline{s}}), \text{ all } t \in u. \tag{4.3}$$

For the special case cop = min, note the reductions of (4.2) and (4.3)

$$(\times_{min}(\underline{a}))_t = \underset{j=1}{\overset{n}{\times}} a^{(j)}_t; \ (\cap_{min}(\underline{a}))_t = \bigcap_{j=1}^{n} a^{(j)}_t, \quad \text{all } t \in u. \tag{4.4}$$

(2) cartesian sum of $\underline{a}$ wrt cocop $\overset{d}{=} \dagger_{cocop}(\underline{a}) \in \text{Flou}(\times\underline{X})$, (4.5)

where analogous to the cartesian product case in (4.2),

$$(\dagger_{cocop}(\underline{a}))_t = \bigcup_{\substack{\text{over all } \underline{s} \in u^n, \\ cocop(\underline{s})=t}} (\times a_{\underline{s}}), \quad \text{for all } t \in u. \tag{4.6}$$

Union becomes for $X = X_1 = .. = X_n$,

$$\cup_{cocop}(\underline{a}) \in \mathrm{Flou}(X); (\cup_{cocop}(\underline{a}))_t = \bigcup_{\substack{\text{over all } \underline{s} \in u^n, \\ cocop(\underline{s})=t}} (\cup\, \underline{a}_{\underline{s}}),\ t \in u. \tag{4.7}$$

For the special case cocop = max, note the reductions of (4.6) and (4.7)

$$(\dagger_{max}(\underline{a}))_t = \mathop{\dagger}_{j=1}^{n} a_t^{(j)} \overset{d}{=} (\mathop{\times}_{j=1}^{n} a_t^{(j)\prime})'; (\cup_{max}(\underline{a}))_t = \bigcup_{j=1}^{n} a_t^{(j)},\ t \in u. \tag{4.8}$$

(3) For any $a^{(j)} \in \mathrm{Flou}(X)$, $j = 1, 2$, it easily follows that

$$a^{(1)} \le a^{(2)} \text{ iff } \phi(a^{(1)}) \le \phi(a^{(2)}) \text{ over } X \text{ iff } a^{(1)} \subseteq a^{(2)}. \tag{4.9}$$

(4) $a' \in \mathrm{Flou}(X)$ is given by , for all $t \in u$,

$$a_t' = (\phi^{-1}(\phi(a)'))_t = (1 - \phi(a))^{-1}[t, 1] = \{x : x \in X \ \&\ \phi(a)(x) \le 1 - t\}$$

$$= X - \phi(a)^{-1}(1 - t, 1] = X - a_{(1-t)^+} \tag{4.10}$$

where

$$a_{(1-t)^+} = \phi(a)^{-1}(1 - t, 1] = \bigcup_{1-t<s\le 1} \phi(a)^{-1}[s, 1] = \bigcup_{1-t<s\le 1} a_s. \tag{4.11}$$

(5) $T(a) \in \mathrm{Flou}(T(X))$, where for all $t \in u$,

$$(T(a))_t = (\phi^{-1}(T(\phi(a))))_t = (T(\phi(a)))^{-1}[t, 1]$$

$$= \{y : y \in Y \ \&\ \sup\{s : s \in u \ \&\ y \in T(a_s)\} \ge t\}. \tag{4.12}$$

Define

$$\mathrm{T}(a) \overset{d}{=} (T(a_t))_{t\in u}. \tag{4.13}$$

Now, $\mathrm{T}(a) \in \mathrm{Flou}(T(X))$. *Proof*: First, the left hand side of (2.4) can be verified directly. As for the right hand side of (2.4): Let $J \subseteq u$, $y \in \bigcap_{s\in J} T(a_s)$. Hence, $y = T(x)$ for some $x \in a_s$, all $s \cup J$, implying $y \in T(\bigcap_{s\in J} a_s) = T(a_{\sup(J)})$, using r.h.s. (2.4) property of a itself. Conversely, if $y \in T(a_{\sup(J)})$, there exists $x \in a_{\sup(J)}$ with $y = T(x)$. But, $a_{\sup(J)} = \bigcap_{s\in J} a_s$, so that $y = T(x)$, $x \in a_s$, all $s \in J$, implying $y \in \bigcap_{s\in J} T(a_s)$. Hence, r.h.s. (2.4) holds and thus $\mathrm{T}(a) \in \mathrm{Flou}(X)$.

Next, applying Theorem 2.1 to $T(a)$, shows for all $t \in u$,

$$(T(a))_t = \{y : y \in Y \ \&\ \phi(T(a))(y) \ge t\} = (\phi(T(a)))^{-1}[t, 1] = T(a_t), \tag{4.14}$$

i.e., using (4.13),

$$T(a) = \mathrm{T}(a). \tag{4.15}$$

In particular, the multiargument case where $X = \times\underline{X}$ becomes

$$T_{cop}(\underline{a}) = T(\times_{cop}(\underline{a})). \tag{4.16}$$

(6) For all $t \in u$,

$$(T^{-1}(b))_t = (\phi^{-1}(T^{-1}(\phi(b)))_t = (\phi(b)_\circ T)^{-1}[t, 1] = T^{-1}(\phi(b)^{-1}[t, 1]) = T^{-1}(b_t). \tag{4.17}$$

(4.17) shows

$$T^{-1}(b) = T^{-1}(b) = (T^{-1}(b_t))_{t\in u}. \tag{4.18}$$

(7) The H-modifier of a is determined as

$$H_\circ a = \phi^{-1}(H_\circ \phi(a)) \in \mathrm{Flou}(X), \tag{4.19}$$

where for all $t \in u$,

$$(H_\circ a)_t = (\phi^{-1}(H_\circ \phi(a)))_t = (H_\circ \phi(a))^{-1}[t, 1] = (H_\circ \phi(a))^{-1}[t, 1] = \phi(a)^{-1}(H^{-1}[t, 1]). \tag{4.20}$$

If H is monotone increasing with $H(0) = 0$ and $H(1) = 1$, then (4.20) becomes

$$(H_\circ a)_t = \phi(a)^{-1}[H^{-1}(t), 1] = a_{H^{-1}(t)}, \quad \text{all } t \in u, \tag{4.21}$$

whence

$$H_\circ a = a_{H^{-1}}. \tag{4.22}$$

On the other hand, if H is monotone decreasing with $H(0)$ 1 and $H(1) = 0$, then (4.20) becomes

$$(H_\circ a)_t = \phi(a)^{-1}[0, H^{-1}(t)] = X \dashv \phi(a)^{-1}(H^{-1}(t), 1] = X \dashv a_{H^{-1}(t)^+}. \tag{4.23}$$

Summarizing the above results:

Theorem 4.1. Let * refer to any of the seven types of operations and relations defined for Mem(X) (or Mem(×X)) in section 3, eqs. (3.4)-(3.20). Let $\phi^{-1}(*)$ refer to the corresponding seven types of operations and relations given for Flou(X) (or related spaces) in this section, eqs. (4.1)-(4.23). Then (using the X form for generality), $\phi : (\mathrm{Flou}(X); \phi^{-1}(*)) \to (\mathrm{Mem}(X); *)$ is a surjective isomorphism.

Proof: Immediate consequence of the constructive procedure of section 3 for τ^{-1} replaced by ϕ (and τ by ϕ^{-1}), relative to the bijection ϕ as shown in Theorem 2.1.

■

In another direction, recall the concept of the sup norm of a fuzzy set membership function (see e.g. Goodman & Nguyen (1985, section 3.3)):

$$\| \ \| : \mathrm{Mem}(X) \to u \ ; \ \|f\| \overset{d}{=} \sup_{x\in X} f(x). \tag{4.24}$$

Then,

$$\|\phi^{-1}(f)\| \overset{d}{=} \sup\{t : t \in u \ \& \ \phi^{-1}(f)_t = f^{-1}[t, 1]$$

$= \sup\{t : t \in u \;\&\; t \le f(x), \text{ for some } x \in X\} = \|f\|, \text{ all } f \in \mathrm{Mem}(X),$ (4.25)

so that

$$\|\phi(a)\| = \|a\|, \text{ all } a \in \mathrm{Flou}(X), \tag{4.26}$$

showing the invariance of $\|\;\|$ wrt ϕ. Similar remarks hold for trace norms, where a fuzzy intersection relative to a fixed membership function is used.

As a final segment to this section, suppose we restrict $\mathrm{Mem}(X)$ to $\mathrm{Dist}(\mathbb{R})$, the class of all cumulative probability distribution functions (cdf's) over the real line $\mathbb{R}$ (recalling that a cdf F is characterized as $F : \mathbb{R} \to u$ being nondecreasing, continuous from the right with $F(-\infty) = 0$ and $F(+\infty) = 1$). Also, define $\mathrm{Ant}(\mathbb{R})$ as the class of all anti-distribution functions G over u in the sense that $G : u \to \mathbb{R}$ is any nondecreasing, continuous from the left function with (abusing notation relative to the domains of use) $G(0) = -\infty$ and $G(1) = +\infty$. Also, recall the pseudoinverse of cdf F as given by

$$F^{\square}(t) \overset{d}{=} \inf F^{-1}[t, 1], \text{ all } t \in u, \tag{4.27}$$

with the usual properties such as $F_o F^{\square}{}_o F = F$ and $F^{\square}{}_o F_o F^{\square} = F^{\square}$, etc. (See e.g., Goodman & Nguyen (1985, pp. 121 et passim).) Dually, define for each $G \in \mathrm{Ant}(\mathbb{R})$, G^{Δ} and $\tau(G)$, where

$$G^{\Delta}(x) \overset{d}{=} \sup G^{-1}(-\infty, x], \text{ all } x \in X;\; \tau(G) \overset{d}{=} ([G(s), +\infty))_{s \in u}, \tag{4.28}$$

and let the $\mathrm{range}(\tau)$ be denoted as $\mathrm{Pseu}(\mathbb{R})$. Then, it follows that for all $F \in \mathrm{Dist}(\mathbb{R})$, $G \in \mathrm{Ant}(\mathbb{R})$,

$$F^{\square\Delta} = F \;;\; G^{\Delta\square} = G\,, \tag{4.29}$$

and hence $(\;)^{\Delta} : \mathrm{Ant}(\mathbb{R}) \to \mathrm{Dist}(\mathbb{R})$ and $(\;)^{\square} : \mathrm{Dist}(\mathbb{R}) \to \mathrm{Ant}(\mathbb{R})$ are well-defined inverse bijections of each other. It also follows for any $F \in \mathrm{Dist}(\mathbb{R})$ that

$$\phi^{-1}(F) = ((\phi^{-1}(F))_t)_{t \in u}\,;\; (\phi^{-1}(F))_t = F^{-1}[t, 1] = [F^{\square}(t), +\infty) = (\tau(F^{\square}))(t), \tag{4.30}$$

and for any $G \in \mathrm{Ant}(\mathbb{R})$, $\qquad \phi(\tau(G)) = G^{\Delta}.$ (4.31)

The above can all be summarized by the following diagram of bijections:

$$\begin{tikzcd}
\mathrm{Flou}(\mathbb{R}) \arrow[rr, "\phi"] & & \mathrm{Mem}(\mathbb{R}) \\
\cup & & \cup \\
\mathrm{Pseu}(\mathbb{R}) \arrow[rr, "\phi"] & & \mathrm{Dist}(\mathbb{R}) \arrow[dl, "(\;)^{\square}"] \\
& \mathrm{Ant}(\mathbb{R}) \arrow[ul, "\tau"] &
\end{tikzcd} \qquad \phi^{-1} = \tau_{\,o}(\;)^{\square}$$

Figure 4.1. Summary of bijections involving cdf's as membership functions

All of the above can be generalized to $\mathbb{R}^n$ with suitable modifications. In addition, the construction technique of section 3, as applied in the earlier part of this section to developing the bijections among Mem(X) and Flou(X) into isomorphisms is valid here as a special case, showing a basic connection between probability (via cdf's) and fuzzy sets.

5. Construction of Operations over Partitioning Spaces Isomorphic to Those over Fuzzy Set Membership Function Spaces.

In addition to the previous notation introduced, denote $\underline{q} \overset{d}{=} (q^{(1)},\ldots,q^{(n)})$ where $q^{(j)} \in \mathrm{Part}(X_j)$ is arbitrary, $j = 1,\ldots,n$. Similarly, denote

$(\phi\circ\psi)(\underline{q}) \overset{d}{=} (\phi(\psi(q^{(1)})),\ldots,\phi(\psi(q^{(n)})))$, noting $\underline{q} \in \mathrm{Part}(\underline{X}) \overset{d}{=} (\mathrm{Part}(X_1),\ldots,\mathrm{Part}(X_n))$ while $(\phi\circ\psi)(\underline{q}) \in \mathrm{Mem}(\underline{X})$, etc.

By use of the isomorphism construction technique discussed in section 3, where now $\tau^{-1} = \phi\circ\psi$ and X is replaced by Mem(X), while Y is replaced by Part(X), the following counterparts are obtained for the seven basic membership operations and relations:

(1) cartesian product of $\underline{q}$ wrt cop $\overset{d}{=} \times_{cop}(\underline{q}) \in \mathrm{Part}(\times\underline{X})$, (5.1)
where

$$\times_{cop}(\underline{q}) = (\phi\circ\psi)^{-1}(\phi\circ\psi)^{-1}(\times_{cop}(\phi\circ\psi)(\underline{q})) \tag{5.2}$$

with index set

$$J_{\times_{cop}(\underline{q})} = \mathrm{range}(\times_{cop}(\phi\circ\psi)(\underline{q})) = \mathop{\times_{cop}}_{j=1}^{n}(J_{q^{(j)}}). \tag{5.3}$$

For each $t \in u$,

$$(\dagger_{cocop}(\underline{q}))_t = ((\phi\circ\psi)^{-1}(\dagger_{cocop}((\phi\circ\psi)(\underline{q}))))_t = \bigcup_{\substack{\text{over all} \\ \underline{s}\in J_{\dagger_{cocop}(\underline{q})}, \\ cocop(\underline{s})=t}} (\mathop{\times}_{j=1}^{n} q^{(j)}_{s_j}) \tag{5.4}$$

with similar forms holding for "intersections".

(2) cartesian sum of $\underline{q}$ wrt cocop $\overset{d}{=} \dagger_{cocop}(\underline{q}) \in \mathrm{Part}(\times\underline{X})$ (5 5)
has index set

$$J_{\dagger_{cocop}(\underline{q})} = \overset{n}{\underset{j=1}{\dagger}}{}_{cocop}(J_{q}(j)). \tag{5.6}$$

For all $t \in u$,

$$(\dagger_{cocop}(\underline{q}))_t = ((\phi_\circ\psi)^{-1}(\dagger_{cocop}((\phi_\circ\psi)(\underline{q}))))_t = \bigcup_{\substack{\text{over all} \\ \underline{s}\in J_{\dagger_{cocop}(\underline{q})}, \\ cocop(\underline{s})=t}} (\overset{n}{\underset{j=1}{\times}} q^{(j)}_{s_j}), \tag{5.7}$$

with similar forms holding for "unions".

(3) For any $q^{(1)}, q^{(2)} \in \text{Part}(X)$, $q^{(1)} \le q^{(2)}$

$$\text{iff } (\phi_\circ\psi)_{\text{over } X}(q^{(1)}) \le (\phi_\circ\psi)(q^{(2)}). \tag{5.8}$$

Then it can be shown that

$$\left.\begin{array}{l} q^{(1)} \le q^{(2)} \text{ iff } q^{(2)} \text{ is a refinement of } q^{(1)}\text{, i.e., for each } s\in J_{q^{(1)}} \\ \text{there exists } I_s \subsetneq J_{q^{(2)}} \text{ with } s \le I_s \;\&\; q^{(1)}_s = \bigcup_{t\in I_s} q^{(2)}_t. \end{array}\right\} \tag{5.9}$$

(4) For all $q \in \text{Part}(X)$, $q' = (\phi_\circ\psi)^{-1}(((\phi_\circ\psi)(q))')$, (5.10)
with index set

$$J_{q'} = \text{rang}(((\phi_\circ\psi)(q))') = 1 - J_q = \{1 - t : t \in J_q\}. \tag{5.11}$$

For all $t \in u$

$$(q')_t = (((\phi_\circ\psi)(q))^{-1}(t))'$$
$$= \{x : x \in X \;\&\; f_q(x) = 1 - t\}\} = f_q^{-1}(1 - t) = q_{1-t}. \tag{5.12}$$

(5) T-attribute transform of $q = T(q) = (\phi_\circ\psi)^{-1}(T((\phi_\circ\psi)(q))) \in \text{Part}(T(X))$ (5.13)
with index set

$$J_{T(q)} = \text{range}(T((\phi_\circ\psi)(q)))$$
$$= \text{range }(T(f_q)) = \{\sup\{s : s \in J_q \;\&\; y \in T(q_s)\} : y \in Y\}, \tag{5.14}$$

For all $t \in u$,

$$(T(q))_t = (T((\phi_\circ\psi)(q))^{-1}(t) = (T(f_q))^{-1}(t)$$
$$= \{y : y \in Y \;\&\; \sup\{s : s \in J_q, y \in T(q_s)\} = t\} \tag{5.15}$$

(6) T^{-1}-attribute transform of $q = (T^{-1}(q))_t = (\phi\circ\psi)^{-1}(T^{-1}((\phi\circ\psi)(q)))$,

(5.16)

with index set

$$J_{T^{-1}(q)} = \text{range}(T^{-1}((\phi\circ\psi)(q))) = \text{range}(T^{-1}(f_q)) = \text{range}(f_q\circ T)$$
$$= \{s : s \in J_q \;\&\; T^{-1}(q_s) \neq \emptyset\}.$$

(5.17)

For all $t \in u$,

$$(T^{-1}(q))_t = ((\phi\circ\psi)^{-1}(T^{-1}((\phi\circ\psi)(q))))_t$$
$$= (T^{-1}(f_q))^{-1}(t) = (f_q\circ T)^{-1}(t)$$
$$= T^{-1}(f_q^{-1}(t)) = T^{-1}(q_t).$$

(5.18)

(7) H-modifier for $q = H\circ q = (\phi\circ\psi)^{-1}(H\circ f_q)$,

(5.19)

with index set

$$J_{H\circ q} = \text{range}(H\circ f_q) = H(J_q).$$

(5.20)

For all $t \in u$,

$$(H\circ q)_t = ((\phi\circ\psi)^{-1}(H\circ f_q))_t = (H\circ f_q)^{-1}(t) = f_q^{-1}(H^{-1}(t)) = q_{H^{-1}(t)} = \bigcup_{s\in H^{-1}(t)\cap J_q} (q_s).$$

(5.21)

Summarizing the above results:

Theorem 5.1. Let $*$ refer to any of the seven types of operations and relations defined for Mem(X) (or Mem($\times\underline{X}$)) in section 3, eqs. (3.4)-(3.20). Let $(\phi\circ\psi)^{-1}(*)$ refer to the corresponding seven types of operations and relations given for Part(X) (or related spaces) in this section, eqs. (5.1)-(5.21). Then (using the X form for generality), $\phi\circ\psi : (\text{Part}(X); (\phi\circ\psi)^{-1}(*)) \to \text{Mem}(X); *)$ is a surjective isomorphism.

Proof. Immediate consequence of the constructive procedure of section 3 for τ^{-1} replaced by $\phi\circ\psi$ (and τ by $(\phi\circ\psi)^{-1}$), relative to the bijection $\phi\circ\psi$ as shown in Theorem 2.2. ∎

<u>Remarks</u> In summary, the following diagram holds, superseding Figure 2.1:

$$\begin{tikzcd} & (\mathrm{Flou}(X)\,;\,\phi^{-1}(*)) \arrow[dr, "\phi\ \mathrm{iso}"] & \\ (\mathrm{Part}(X);\,(\phi\circ\psi)^{-1}(*)) \arrow[ur, "\mathrm{iso}\ \psi"] \arrow[rr, "\mathrm{iso}", "\phi(X)\circ\psi(X)"'] & & (\mathrm{Mem}(X):\,*) \end{tikzcd}$$

Figure 5.1. Summary of isomorphisms among Mem(X), Flou(X), Part(X).

Thus, the initial Zadeh operations and relations defined over Mem(X), the usual semantically or numerically-oriented space representing fuzzy set membership functions, can be isomorphically represented by *both* counterparts over Flou(X) and those over Part(X). The last two spaces in light of Theorems 4.1 and 5.1 can be considered to be the natural syntactic or algebraic structures representing fuzzy sets. Of course, a number of other operations and relations could have been considered, but the above seven seems to be a reasonable demonstration of the natural relations among the spaces. (Conditioning will be treated later as a special type of operation.)

The next section considers the important special case of finite-valued membership functions and the corresponding flou and partitioning sets, together with some relationships with conditional events, as previously developed for ordinary (i.e., non-fuzzy) events and sets.

6. Finite-Valued Fuzzy Set Membership Functions and Relations with Conditional and Unconditional Sets

In this section we specialize some of the previous results for the general case to the setting where only finite-valued membership functions are considered and relate this to conditional event algebra for the three-valued subcase.

In particular, let $f \in \mathrm{Mem}(X)$ be such that it is arbitrary fixed with

$$\mathrm{range}(f) = \{t_j : j = 1,\ldots,m\}\ ;\quad 0 \le t_1 < t_2 < \ldots < t_m \le 1, \tag{6.1}$$

for some arbitrarily fixed positive integer m and real t_j. It follows that the corresponding flou set is from Theorem 2.1

$$\phi^{-1}(f) = ((\phi^{-1}(f))_s)_{s \in u} \tag{6.2}$$

where now

$$(\phi^{-1}(f))_s = f^{-1}[s,1] = \begin{cases} \emptyset, \text{ if } t_m < s \le 1; \\ f^{-1}\{t_{j+1}, t_{j+2}, \ldots, t_m\} \\ f^{-1}(t_{j+1}) \cup f^{-1}(t_{j+2}) \cup \ldots \cup f^{-1}(t_m), \\ \text{if } t_j < s \le t_{j+1},\ j=1,2,\ldots,m-1; \\ X, \text{ if } 0 \le s \le t_1. \end{cases} \tag{6.3}$$

The corresponding partitioning set is from Theorem 2.2

$$(\phi_\circ\psi)^{-1}(f) = (((\phi_\circ\psi)^{-1}(F))_s)_{s\in J_{(\phi_\circ\psi)^{-1}(f)}} \tag{6.4}$$

where from (2.37)

$$((\phi_\circ\psi)^{-1}(f))_s = f^{-1}(s),\ s \in J_{(\phi_\circ\psi)^{-1}(f)}, \tag{6.5}$$

where index set

$$J_{(\phi_\circ\psi)^{-1}(f)} = \text{range}(f) \tag{6.6}$$

given in (6.1).

It is clear by inspection that any finite partitioning $q = (q_s)_{s\in J_q} \in \text{Par}(X)$ arises from some finite-valued f. (See also the proof of Theorem 2.2.) Similar remarks hold for the correspondences of finite flou sets, i.e. flou sets with only a finite number of distinct component sets, and finite-valued membership functions. Summarizing:

Theorem 6.0. Theorems 2.1 and 2.2 remain valid when the bijections ϕ, ψ, and $\phi_\circ\psi$ are all restricted to the classes of finite-valued elements -- in the above senses -- of their domains. Indeed, in light of Theorems 4.1 and 5.1, these bijections are actually isomorphisms when so restricted. ∎

Next, let us treat in some detail two particular subcases of finite-valued membership functions and a modified third subcase.

First, consider single-valued, or equivalently, *constant*, membership functions and their corresponding flou and partitioning sets: For any constant c in u, use the standard identification with $c : X \to u$, where

$$c(x) = c(\text{constant}), \text{ all } x \in X, \tag{6.7}$$

is used. Denote the class of all such functions as

$$\text{Mem}_1(X) \overset{d}{=} \{c : c : X \to u,\ c \in u\}. \tag{6.8}$$

The corresponding flou set is easily seen to be

$$\phi^{-1}(c) = ((\phi^{-1}(c))_s)_{s \in u}, \tag{6.9}$$

where for all $s \in u$,

$$(\phi_{-1}(c))_s = c^{-1}[s, 1] = \begin{cases} X, & \text{if } 0 \le s \le c, \\ \emptyset, & \text{if } c < s \le 1. \end{cases} \tag{6.10}$$

Denote the class of all such flou sets as

$$\mathrm{Flou}_1(X) \overset{d}{=} \{\phi^{-1}(c) : c \in u\}. \tag{6.11}$$

Next, the corresponding partitioning set to c is

$$(\phi_\circ\psi)^{-1}(c) = (((\phi_\circ\psi)^{-1}(c))_s)_{s \in J_{(\phi_\circ\psi)^{-1}(c)}} \tag{6.12}$$

where

$$J_{(\phi_\circ\psi)^{-1}(c)} = \mathrm{range}(c) = \{c\};\ ((\phi_\circ\psi)^{-1}(c))_c = c^{-1}(c) = X. \tag{6.13}$$

That is,

$$(\phi_\circ\psi)^{-1}(c) = \{X\} \text{ (with index value c)}. \tag{6.14}$$

Denote the class of all such partitioning sets as

$$\mathrm{Part}_1(X) \overset{d}{=} \{(\phi_\circ\psi)^{-1}(c) : c \in u\}. \tag{6.15}$$

Next, consider membership functions which can have possibly two values 0 or 1, i.e., the class of all ordinary set membership, or equivalently, *indicator*, functions $1_A : X \to \{0, 1\} \in \mathrm{Mem}(X)$, where the standard relation holds for any ordinary subset A of X

$$1_A(x) \overset{d}{=} \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in A'. \end{cases} \tag{6.16}$$

Corresponding to any 1_A, $A \in \mathscr{P}(X)$, the flou and partitioning sets are:

$$\phi^{-1}(1_A) = ((\phi^{-1}(1_A))_s)_{s \in u}, \tag{6.17}$$

where

$$(\phi^{-1}(1_A))_s = 1_A^{-1}[s, 1] = \begin{cases} X, & \text{if } s = 0 \\ A, & \text{if } 0 < s \le 1. \end{cases} \tag{6.18}$$

$$(\phi_\circ\psi)^{-1}(1_A) = (((\phi_\circ\psi)^{-1}(1_A))_s)_{s \in J_{(\phi_\circ\psi)^{-1}(1_A)}}; \tag{6.19}$$

index set

$$J_{(\phi_\circ\psi)^{-1}(1_A)} = \mathrm{range}(1_A) = \{0, 1\}, \tag{6.20}$$

unless

$$A = X, \text{ implying } J_{(\phi\circ\psi)^{-1}(1_X)} = \{1\}; \text{ or } A = \emptyset, \text{ implying } J_{(\phi\circ\psi)^{-1}(1_\emptyset)} = \{0\}. \tag{6.21}$$

For $A \in \mathcal{P}(X)$ in general again,

$$((\phi\circ\psi)^{-1}(1_A))_0 = 1_A^{-1}(0) = A' ; \; ((\phi\circ\psi)^{-1}(1_A))_1 = 1_A^{-1}(1) = A. \tag{6.22}$$

The special cases $A = X$ and $A = \emptyset$ yield

$$(\phi\circ\psi)^{-1}(1_X) = \{X\} \text{ (with index val. 1); } (\phi\circ\psi)^{-1}(1_\emptyset) = \{X\} \text{ (with index val. 0).} \tag{6.23}$$

Denote the above class of membership functions with values in $\{0, 1\}$ as $M_2(X)$ with the corresponding flou class as $Flou_2(X)$ and the corresponding partitioning set class as $Part_2(X)$.

Next, consider any fixed $t \in u$ and define the class

$$Mem_{t,3}(X) \overset{d}{=} \{0, t, 1\}^X = \{f : f \in Mem(X) \;\&\; range(f) \subseteq \{0, t, 1\}\}. \tag{6.24}$$

In turn, define the union

$$Mem_3(X) \overset{d}{=} \bigcup_{t\in u} Mem_{t,3}(X), \tag{6.25}$$

noting from (6.24),

$$Mem_{0,3}(X) = Mem_{1,3}(X) = \{0, 1\}^X = Mem_2(X). \tag{6.26}$$

For any $t \in u$ and any $f_t \in Mem_{t,3}(X)$, the corresponding flou and partitioning sets are:

$$\phi^{-1}(f_t) = ((\phi^{-1}(f_t))_s)_{s\in u}, \tag{6.27}$$

$$(\phi^{-1}(f_t))_s = f_t^{-1}[s, 1] = \begin{cases} X, \text{ if } s = 0 \\ f_t^{-1}(t) \cup f_t^{-1}(1), \text{ if } 0 < s \le t, \\ f_t^{-1}(1), \text{ if } s = 1. \end{cases} \tag{6.28}$$

$$(\phi\circ\psi)^{-1}(f_t) = (((\phi\circ\psi)^{-1}(F_t))_s)_{s\in J_{(\phi\circ\psi)^{-1}(f_t)}}, \tag{6.29}$$

with index set

$$J_{(\phi\circ\psi)^{-1}(f_t)} = range(f_t) = \{0, t, 1\}, \tag{6.30}$$

and for $s \in \{0, t, 1\}$

$$((\phi\circ\psi)^{-1}(f_t))_s = f_t^{-1}(s) = \begin{cases} f_t^{-1}(0), & \text{if } s = 0 \\ f_t^{-1}(t), & \text{if } s = t \\ f_t^{-1}(1), & \text{if } s = 1. \end{cases} \tag{6.31}$$

Next, let $q = (q_0, q_t, q_1)$ be any ordered partitioning of X where any one or two of the component q_s may possibly be vacuous. Denote $\text{Part}^{(3)}(X)$ as the class of all such ordered partitionings of X. In turn, for each $q \in \text{Part}^{(3)}(X)$, define the class

$a_t(q) \overset{d}{=} \{f : f \in \text{Mem}_3(X)$ & for all $s \in \{0, t, 1\}$, if $q_s \neq \emptyset$, then $f(x) = s$, for all $x \in q_s$; $t \in u\}$. (6.32)

In a related direction, for any sets $A, B \in \mathcal{P}(X)$, and any $t \in u$, define one natural extension of the ordinary set indicator function given in eq. (6.16) to three values as (using $\vee$ for max, $\wedge$ for min, etc.) $1_{(A|B)_t}$,

$$1_{(A|B)_t}(x) \overset{d}{=} 1_{A\cap B}(x) \vee 1_{B'}(x)\cdot t = \begin{cases} 1, & \text{if } x \in A \cap B, \\ 0, & \text{if } x \in B - A, \\ u, & \text{if } x \in B'. \end{cases} \tag{6.33}$$

Finally, define the function $1_{(A|B)}$ as

$$1_{(A|B)}(x) \overset{d}{=} 1_{A\cap B}(x) \vee 1_{B'}(x)\cdot u = \begin{cases} 1, & \text{if } x \in A \cap B, \\ 0, & \text{if } x \in B - A, \\ u, & \text{if } x \in B'. \end{cases} \tag{6.34}$$

$1_{(A|B)}$ is the standard *conditional event* (or conditional set) *indicator function*, as first developed independently by Schay (1968) and DeFinetti (1974). More on this topic later; summarizing the above relations:

Theorem 6.1. The following relations hold among the special cases of Mem(X), Flou(X), and Part(X) considered above:

(i)
$$\left.\begin{matrix} \text{Mem}_1(X) \\ \text{Mem}_2(X) \subseteq \text{Mem}_{t,3}(X) \end{matrix}\right\} \subseteq \text{Mem}_3(X) \subseteq \text{Mem}(X), \tag{6.35}$$

with the same relations holding in (6.35) when Mem is replaced by Flou and Part.

(ii) $(\text{Part}_j(X), (\phi\circ\psi)^{-1}(*))$, $(\text{Flou}_j(X); \phi^{-1}(*))$, and $(\text{Mem}_j(X); *)$ are all isomorphic relative to the appropriate restrictions for ψ, ϕ, and $\phi\circ\psi$ as given in Theorems 4.1 and 5.1 and summarized in Figure 5.1, when: j = 1, as given in (6.7)-(6.15); j = 2, as given

in (6.16)-(6.23); and $j = (t, 3)$, as given in (6.24)-(6.31).
(iii) For all $A, B \in P(X)$, one can make the natural identification

$$1_{(A|B)} = \{1_{(A|B)_t} : t \in u\}. \tag{6.36}$$

Since also

$$\mathrm{Mem}_{t,3}(X) = \bigcup_{q \in \mathrm{Part}^{(3)}(X)} \{a_t(q)\}, \tag{6.37}$$

one also has the identifications

$$\mathrm{Mem}_3(X) = \bigcup_{\substack{q \in \mathrm{Part}^{(3)}(X) \\ t \in u}} \{a_t(q)\} = \{1_{(A|B)} : A, B \in \mathcal{P}(X)\}. \tag{6.38}$$

Proof: Straightforward from the definitions. ∎

Brief overview of conditional event indicator functions and conditional events.

With the basic tie-in between conditional even indicator functions and three-valued fuzzy set membership functions pointed out, a short summary of the development of conditional events and their indicator functions will be presented. (See Goodman (1987), Goodman & Nguyen (1988, 1991), and Goodman, Nguyen, Walker (1991) for general background.)

In the following, unconditional events or sets are indicated by $A, B, C, D, ..$ which, in place of the concrete situation (via direct considerations or use of the Stone Representation Theorem), where they are all subsets of X forming a boolean algebra which is a subclass of $P(X)$, one can consider them to form an abstract boolean algebra R or events or propositions. In this case, the operators are: conjunction $\cdot$ (replacing the more concrete $\cap$); disjunction $\vee$ (replacing the more concrete $\cup$); complement or negation $(\,)'$ (which for simplicity is denoted by the same symbol as in the concrete case); $\leq$ (replacing the more concrete $\subseteq$); $<$ (replacing the more concrete $\subset$); 1 (replacing the more concrete X); 0 (replacing the more concrete $\emptyset$); material/logical implication $\Rightarrow$ given as $B \Rightarrow A \overset{d}{=} B' \vee A$ (replacing the more concrete $B' \cap A$); material/logical equivalence $\Leftrightarrow$ given as

$$B \Leftrightarrow A \overset{d}{=} (B \Rightarrow A)\cdot(A \Rightarrow B) = AB \vee A'B' = (A + B)',$$

dropping the conjunction notation when no ambiguity arises, where $A + B \overset{d}{=} A'B \vee AB'$, etc.

Conditional events arise in order to provide a systematic/rigorous way to deal with arbitrary logical combinations of implicative statements relative to all probability evaluations, when it is appropriate to interpret the probability evaluations of each separate implicative statement as a conditional probability in the natural sense. For example, suppose one wants to obtain the probability p((if B then A) or (if D then C)), where the evaluations $p(\text{if } B \text{ then } A) = p(A|B)$ $(=p(AB)/p(B)$, assuming $p(B) > 0)$ and $p(\text{if } D \text{ then } C) = p(C|D)$ hold. No current standard approach exists in the numerically-oriented field of conditional probability (including Renyi's comprehensive extension (1970)) whereby the implicatives "if B then A" and "if D then C" can be given meaning, independent of the particular probability p being used. This is so that these expressions can be combined with other expressions, in conditional or unconditional form, analogous to the way the unconditionals A, B, C, D, .. can all be manipulated and combined, compatible with all probability evaluations. Certainly, a "natural" candidate for such an interpretation is material implication, so that in the above example one would obtain by the usual Poincaré expansion

$$p((\text{if } B \text{ then } A) \text{ or } (\text{if } D \text{ then } C)) = p((B \Rightarrow A) \vee (D \Rightarrow C)) = p(B' \vee A \vee D' \vee C)$$

$$= p(B') + p(A) + p(D') + p(C) - p(B'A) - p(B'D') - p(B'C)$$
$$- p(AD') - p(AC) - p(D'C) + p(B'AD') + p(B'AC) + p(B'D'C)$$
$$+ p(AD'C) - p(B'AD'C). \tag{6.39}$$

However, the main drawback to the above approach is that *material implication is inconsistent with conditional probability as its probability evaluation* since it can be readily shown: [Author's note: this and all subsequent results can be found in the above reference Goodman, Nguyen, Walker (1991) or in Goodman (1991) in detail; for the most part, these references will not be repeated here.]

$$p(B \Rightarrow A) = a - p(B) + p(AB) = p(A|B) + (p(A'|B) \cdot p(B')) \geq p(A|B), \tag{6.40}$$

provided $p(B) > 0$, where in general strict inequality holds above. Indeed, Lewis (1976) showed that in general *there is no function* $g : R^2 \to R$ *(boolean or otherwise!) such that equality could hold in a modified version of (6.40), where* $\Rightarrow$ *is replaced by* g, i.e.,

For all $g : R^2 \to R$, it is not true that $p(g(A, B)) = p(A|B)$, all $A, B \in R$. (6.41)

Thus, the search for syntactic or algebraic interpretations of implicatives compatible with all conditional probability evaluations, if at all possible must lie in functions $g : R^2 \to S$, where $S \nsubseteq R$. Of course, if all of the antecedents of the implicatives present are identical, then no real problem arises and the search for algebraic representations of "conditional events" $g(A, B) = (A|B)$ is avoided. For example, in the original example, if antecedents $B = D$, then it is indeed natural to compute in effect

$$p((\text{if } B \text{ then } A) \text{ or } (\text{if } B \text{ then } C)) = p((A|B0 \vee (C|B)) = p((A \vee C|B)) = p(A \vee C|B), \tag{6.42}$$

provided $p(B) > 0$, where in the standard approach to conditional probability, the middle two expressions would not be used. However, when the antecedents are not all identical, in general it would seem that one should seek a common denominator-like antecedent so that the technique provided through the example in (6.42) could be employed. It will be seen later that this is an equivalent viable approach to the basic problem, but that the "common denominator" is not trivial.

While it was stated previously that the direction of conditional probability is away from the algebraic, a relative handful of researchers have seriously considered this problem at one time or another. This list includes: Boole (1854, Chpt. et passim), Hailperin's restatement and rigorizing of Boole's ideas using the modern approach of Chevalley-Uzkov algebraic fractions; Mazurkiewicz' original use of principal ideal cosets (in a boolean algebra) to represent conditional events (1956), Copeland's futile attempts (seen now in light of Lewis' "triviality" result cited above)(1950, 1956) at forcing, in effect, conditional events to be in the original boolean algebra R; DeFinetti's efforts, including the defining of conditional event indicator functions (as in (6.34)) (1974) independent of all others; Schay's proposal for conditional event indicator functions (1968), independently coinciding with DeFinetti, but also for the first time, developing a full calculus of operations and relations for conditional events; Adams (1975) proposing operations for conditional events that independently coincided with Schay, but gave no interpretation for the conditional events themselves!; Calabrese (1987), also independently of all others, first proposing that conditional events should be interpreted as partial deduct equivalence classes, and in turn developed as Schay before him, a full calculus of operations and relations coinciding for the most part with Schay's results; and also recently, among others, Bruno & Gilio (1985) bringing forth the basic issue of combining implicatives compatible with conditional probability and proposing, in part, a calculus of operations.

All of this lead the author and colleague (H.T. Nguyen) to inquire if there is any unified approach to the basic problem which does not rely upon ad hoc formulations for both the form conditional events must take as well as their operations extending the usual boolean ones for the unconditional case. Certainly, the indicator function approach of Schay and DeFinetti was plausible, but Schay (being the only one of the pair attempting to develop operations and relations) did not justify the choice of his operations. Similarly, Calabrese provided a rationale for his choice of the structure of conditional events, but other than empirical appeal, none for his operations and relations. The others mentioned in the list above did not attepmt to develop operations among conditional events with differing antecedents, except for Adams' formal proposals previously indicated.

The results of this inquiry lead to the following, which provided a new calculus of operations and relations for conditional events, while at the same time justified and related the previous work of most of those mentioned above:

Call any $g : R^2 \to S$ a *feasible candidate for being a conditional even forming function* iff $S = \text{range}(g)$ and

$$g(A, B) = g(AB, B);\ \text{if } g(A, B) = g(C, D),$$
$$\text{then } AB = CD \ \&\ B = D, \quad \text{for all } A, B, C, D \in R, \tag{6.43}$$

noting that when the above holds, then for all prob. $p : R \to u$,

$$p(g(A, B)) = p(A|B),\ p(B) > 0;\ \text{all } A, B \in R, \tag{6.44}$$

is well-defined. Also, define the *natural mapping* $\text{nat} : R^2 \to R$, where for $A, B \in R$,

$$\text{nat}(A, B) \overset{d}{=} R\cdot B' \vee AB \overset{d}{=} \{x\cdot B' \vee AB : x \in R\}$$
$$= \{y : y \in R \ \&\ AB \le y \le B \Rightarrow A\} = \{y : y \in R \ \&\ yB = AB\}, \tag{6.45}$$

the *principal ideal coset generated by* B' with residue AB, noting that for each fixed B, $\text{nat}(\cdot, B) : R \to R/RB'$ is a homomorphism, where for any $A \in R$,

$$\text{nat}(A, B) \in \text{nat}(\cdot, B)(R) = \{\text{nat}(A, B) : A \in R\} = R/RB', \tag{6.46}$$

the *boolean quotient algebra* with the usual coset operations $\cdot, \vee, (\ )'$

$$\text{nat}(A, B)' = \text{nat}(A', B)\ ;\ \text{nat}(A, B)*\text{nat}(C, B) = \text{nat}(A*C, B),$$
$$* = \cdot, \vee, +, \text{ all } A, B, C \in R. \tag{6.47}$$

Denote

$$\tilde{R} \overset{d}{=} \text{range}(\text{nat}) = \{\text{nat}(A, B) : A, B \in R\} = \bigcup_{B \in R} R/RB' \subseteq \mathcal{P}(R), \tag{6.48}$$

the class of all principal ideal cosets of R.

Theorem 6.2. Structure of conditional events.

(i) nat is a feasible candidate for being a conditional event forming function.

(ii) If $g : R^2 \to S$ is any feasible candidate for being a conditional event forming function, then g is globally isomorphic to nat. That is, there is a bijection $\kappa : S \to \tilde{R}$, where $\kappa \circ g = \text{nat}$ and for each $B \in R$, $\kappa_B : S_B \to R/R\cdot B'$ is a bijection, and hence, an isomorphism through the same technique as in the beginning of section 3 inducing an algebraic structure on S via R/RB', where

$$S_B \overset{d}{=} \text{range}(g(\cdot, B)) = \{g(A, B) : A \in R\} \tag{6.49}$$

and

$$\kappa_B(g(A, B)) \overset{d}{=} \kappa(g(A, B)) = \text{nat}(A, B), \text{ all } A \in R \tag{6.50}$$

■

Remarks. Theorem 6.2 justifies the choice for conditional event forming function to be nat, so that from now on, define

$$(A|B) \overset{d}{=} \text{nat}(A, B), \text{ all } A, B \in R;\ (R|R) \overset{d}{=} \tilde{R} = \{(A|B) : A, B \in R\}, \tag{6.51}$$

and note, via (6.44), any prob. $p : R \to u$ extends consistently to $p : (R|R) \to u$, where

$$p((A|B)) = p(A|B), \text{ all } (A|B) \in (R|R),\ p(B) > 0. \tag{6.51'}$$

It can be shown that the algebraic fraction approach of Hailperin and the partial logical deduct approach of Calabrese, both cited earlier, are, in fact, equivalent to the form nat.

Note the division of conditional events into 5 distinct classes:

(I) *Unconditional events in conditional form*:
Since it follows readily that one can identify

$$(A|1) = A, \text{ all } A \in R, \text{ whence } R \subseteq (R|R) \subseteq \mathscr{P}(R), \tag{6.52}$$

call all such conditional events unconditional ones, noting the probability assignment, via (6.44) becomes here simply

$$p((A|1)) = p(A). \tag{6.53}$$

(II) *The indeterminate conditional event*:

$$(A|0) = (0|0) = R, \text{ all } A \in R, \tag{6.54}$$

noting

$$p((0|0)) \text{ not defined.} \tag{6.55}$$

(III) *Unity-type conditional events*: Call the class of all such events $\mathscr{U}$
For all $B \in R, B \neq 0$, $(1|B) = (B|B) = RB' \vee B = R \vee B = \{x : x \in R \ \&\ x \geq B\}$,

$$\tag{6.56}$$

the *principal filter of* R *generated by* B, noting the probability evaluation

$$p((B|B)) = p(B|B) = 1. \tag{6.57}$$

(IV) *Zero-type conditional events*: Call the class of all such events $\mathscr{Z}$

$$\text{For all } B \in R, B \neq 0, (0|B) = (B'|B) = RB' = \{xB' : x \in R\}, \tag{6.58}$$

the *principal ideal of* R generated by B', noting the probability evaluation

$$p((O|B)) = p((B'|B)) = p(0|B) = 0. \tag{6.59}$$

(V) *Proper conditional events*:

$$\text{For all } 0 < A < B < 1,\ A, B \in R,\ (A|B) = RB' \vee AB, \tag{6.59'}$$

with probability evaluation

$$0 < p((A|B)) = p(A|B) < 1. \tag{6.60}$$

Note also the basic properties for all $(A|B), (C|D) \in (R|R)$, from (6.43):

$$(A|B) = (AB|B) \;\&\; (A|B) = (C|D) \text{ iff } AB = CD \;\&\; B = D. \tag{6.61}$$

Returning to the conditional event indicator function given in (6.34), note that by its very definition and use of eqs. (6.38) and (6.61), where now the concrete case of $R \subseteq P(X)$ holds, it follows that $Mem_3(X)$ (with the modification that $P(X)$ in its characterization in (6.38) is replaced by R) and $(R|R)$ are bijective through the relation

$$1_{(A|B)} \mapsto (A|B), \text{ all } A, B \in R. \tag{6.62}$$

Finally, it should be remarked that the conditional event indicator function takes on the following forms relative to each of the 5 types of conditional events:

(I) Unconditional events: $1_{(A|1)} = 1_A \in Mem_2(X)$. (6.63)

(II) Indeterminate event: $1_{(0|0)} = u(\text{const.}) = Mem_1(X)$ (via (6.36)). (6.64)

(III) Unity-type conditional event: $\text{range}(1_{(B|B)}) = \{u, 1\}, 0 < B < 1$. (6.65)

(IV) Zero-type conditional event: $\text{range}(1_{(0|B)}) = \{0, u\}, 0 < B < 1$. (6.66)

(V) Proper conditional events: $\text{range}(1_{(A|B)}) = \{0, u, 1\}$.

The next theorem motivates the choice of operations and relations over $(R|R)$ to be determined:

Theorem 6.3. Characterization of monotonicity of conditional probabilities, ordering of conditional event indicator functions and zero and unity values.

As before, let $R \subseteq \mathcal{P}(X)$ be a fixed boolean algebra of sets. In addition, suppose (needed only for probability part) R is atomic. For any $(A|B), (C|D) \in (R|R)$, but not indeterminate:

(i) If $(A|B)$ is not zero-type and $(C|D)$ is not unity type (certainly satisfied if both are proper), then the following three statements are equivalent:

(I) $1_{(A|B)} \le 1_{(C|D)}$ point-wise over X.

(II) $AB \le CD \;\&\; C'D \le A'B$ (i.e. $B \Rightarrow A \le D \Rightarrow C$).

(III) For all prob. $p : R \to u$, with $p(B), p(D) > 0$, $p(A|B) \le p(C|D)$.

(ii) $(A|B)$ is of zero-type iff $1_{(A|B)} \le u$ over X (wrt order $0 \le u \le 1$) iff for all prob. $p : R \to u$ with $p(B) > 0, p(A|B) = 0$.

(iii) $(A|B)$ is of unity-type iff $1_{(A|B)} \ge u$ over X iff for all prob. $p : R \to u$ with $p(B) > 0, p(A|B) = 1$. ∎

Consider next the standard *functional image extensions* of an arbitrary function, say,

$f : Y^n \to Z$ to the power class level $\hat{f} : \mathcal{P}(Y)^n \to \mathcal{P}(Z)$, where

$$\text{for all } \mathcal{A}_j \in \mathcal{P}(Y),\ \hat{f}(\mathcal{A}_1,\dots,\mathcal{A}_n) \overset{d}{=} \{f(x_1,\dots,x_n) : x_j \in \mathcal{A}_j,\ j = 1,\dots,n\}. \tag{6.67}$$

If $\mathcal{Q} \subseteq \mathcal{P}(Y)^n$ and $\mathcal{R} \subseteq \mathcal{P}(Z)$ are subclasses of interest, it is important to determine whether the restriction of $\hat{f}$ to $\mathcal{Q}$ is closed wrt to $\mathcal{R}$, i.e., range($\hat{f}$ restrict. to $\mathcal{Q}$) $\subseteq \mathcal{R}$. In particular, for the problem at hand, $Y = Z = R$, f is any n-ary boolean function over $\mathcal{R}$, $\mathcal{Q} = (R|R)^n$, and $\mathcal{R} = (R|R)$. It is fortuitous that in this case closure indeed does hold, as the following theorem states, where for simplicity the hat $(\hat{\ })$ notation is omitted:

Theorem 6.4. Functionally-image extended boolean operations and relations over (R|R).

For all $A, B, C, D, A_j, B_j \in R$, $j = 1,\dots,n$:

(i) All functionally-imaged extended boolean operations over R to being over $(R|R)$ are closed and computable for $n = 1$ and 2 as:

$$(A|B)' \overset{d}{=} \{x' : x \in (A|B)\} = (A'|B);\quad (A|B)\cdot(C|D)$$
$$\overset{d}{=} \{x\cdot y : x \in (A|B),\ y \in (C|D)\} = (ABCD|r_2); \tag{6.68}$$

$$(A|B) \vee (C|D) \overset{d}{=} \{x \vee y : x \in (A|B),\ y \in (C|D)\} = (AB \vee CD|q_2); \tag{6.69}$$

$$(A|B) + (C|D) \overset{d}{=} \{x + y : x \in (A|B),\ y \in (C|D)\} = (AB + CD|s_2); \tag{6.70}$$

$$(C|D) \Rightarrow (A|B) \overset{d}{=} \{y \Rightarrow x : x \in (A|B),\ y \in (C|D)\}$$
$$= (C|D)' \vee (A|B) = (C'D \vee AB|t_2); \tag{6.71}$$

$$(C|D) \Leftrightarrow (A|B) \overset{d}{=} \{y \Leftrightarrow x : x \in (A|B),\ y \in (C|D)\}$$
$$= ((C|D) \Rightarrow (A|B))\cdot((A|B) \Rightarrow (C|D)) = ((A|B) + (C|D))' = (AB \Leftrightarrow CD|s_2), \tag{6.72}$$

where

$$r_2 \overset{d}{=} A'B \vee C'D \vee BD = A'B \vee C'D \vee ABCD;$$
$$q_2 \overset{d}{=} AB \vee CD \vee BD = AB \vee CD \vee A'BC'D;$$
$$s_2 \overset{d}{=} BD;\quad t_2 \overset{d}{=} C'D \vee AB \vee BD = C'D \vee AB \vee A'BCD. \tag{6.73}$$

(ii) Part (i) above can be extended the same way to arbitrary n, yielding the closed forms for $\cdot$, $\vee$, $+$:

$$(A_1|B_1)\cdot..\cdot(A_n|B_n) = (A_1B_1\cdot..\cdot A_nB_n|r_n);\ r_n \overset{d}{=} A_1'B_1 \vee..\vee A_n'B_n \vee (A_1B_1\cdot..\cdot A_nB_n); \tag{6.74}$$

$$(A_1|B_1) \vee..\vee (A_n|B_n) = (A_1B_1 \vee..\vee A_nB_n|q_n);$$
$$q_n \overset{d}{=} A_1B_1 \vee..\vee A_nB_n \vee (A_1'B_1\cdot..\cdot A_n'B_n); \tag{6.75}$$

$$(A_1|B_1) +..+ (A_n|B_n) = (A_1B_1 +..+ A_nB_n|s_n);\quad s_n \overset{d}{=} B_1\cdot..\cdot B_n. \tag{6.76}$$

(iii) Extend the natural (partial, indeed, lattice) order $\leq$ over R to (R|R) by defining analogous to the case for R,

$$(A|B) \leq (C|D) \text{ iff } (A|B) = (A|B)\cdot(C|D). \tag{6.77}$$

Then, it can be shown

$$(A|B) \leq (C|D) \text{ iff } (C|D) = (A|B) \vee (C|D) \text{ iff } AB \leq CD \;\&\; C'D \leq A'B$$
$$\text{iff } AB \leq CD \;\&\; B \Rightarrow A \leq D \Rightarrow C. \tag{6.78}$$

(iv) Some miscellaneous properties:

Chaining: $(A|B)\cdot B = AB;\ (A|BC)\cdot(C|B) = (AC|B);$ (6.79)

Bayes' Theorem: If $A_1 \vee..\vee A_n \geq B$, then $(A_j|B)=((B|A_j)\cdot A_j|\bigvee_{j=1}^{n}((B|A_j)\cdot A_j));$ (6.80)

$$C \vee (A|B) = (C \vee A|C \vee B)\ ; \qquad (A|B) = (CA|C \Rightarrow A)\ ;$$
$$(A|B) = (AB \vee B'\cdot(0|0))\ ;\quad (R|R) = R \vee (R\ (0|0))\ ; \tag{6.81}$$
$$(B|B) = B \vee (0|0)\ ;\quad (0|B) = B'\cdot(0|0)\ ;$$
$$\mathscr{U}= (R \dashv \{0\}) \vee (0|0)\ ;\quad \mathscr{Z}= (R \dashv \{1\})\cdot(0|0)\ . \tag{6.82}$$

Equal antecedent/reduction to coset operations:

$$(A_1|B) *..* (A_n|B) = (A_1 *..* A_n|B),\ * = \cdot, \vee, +. \tag{6.83}$$

Remarks.

(i) Theorem 6.4 shows that any n-ary boolean function over (R|R) is not only closed but is feasible to compute in terms of the antecedent and consequent consisting of ordinary unconditional boolean operations. Thus, one evaluates any arbitrary combination of conditional or unconditional events (remembering unconditional events are conditional ones with 1 in the antecedent) for a given probability measure $p : R \to u$ as

$$p(\text{comb}((A_1|B_1),...,(A_n|B_n)))$$
$$= p((\text{comb}_1(A_1B_1, B_1,..,A_nB_n, B_n)|\text{comb}_2(A_1B_1, B_1,...,A_nB_n, B_n))),\ \text{by Thm 6.4}$$
$$= p(\text{comb}_1(A_1B_1, B_1,..,A_nB_n, B_n)|\text{comb}_2(A_1B_1, B_1, ..,A_nB_n, B_n)),\ \text{by (6.51')} \tag{6.84}$$

finally obtained by the usual rules for conditional probability and boolean algebra expansions.

Thus, the original example addressed by material implication in (6.39) becomes

$$\begin{aligned} p((\text{if } B \text{ then } A) \text{ or } (\text{if } D \text{ then } C)) &= p((A|B) \vee (C|D)) \\ &= p((AB \vee /cd|AB \vee CD \vee A'BC'D)) \\ &= p(AB \vee CD|AB \vee CD \vee AA'BC'D) \\ &= p(AB \vee CD)/(p(AB \vee CD) + p(A'BC'D)), \text{ etc.} \end{aligned} \quad (6.85)$$

(ii) Returning to Theorem 6.3 (i), it follows immediately that Theorem 6.4 (iii) (eq. (6.78)) shows the basic compatibility of partial order $\leq$ over $(R|R)$ relative to monotonicity of probability and partial ordering of conditional event indicator functions: For any $(A|B)$, $(C|D) \in (R|R)$ not indeterminate with $(A|B)$ not zero-type and $(C|D)$ not unity type, the following statements are equivalent for R assumed atomic:

(I) $1_{(A|B)} \leq 1_{(C|D)}$ point-wise over X.

(II) $(A|B) \leq (C|D)$.

(III) $p(A|B) \leq p(C|D)$, all prob. $p : R \to u$, with $p(B), p(D) > 0$.

(iii) Theorem 6.4 can also be used to show that the algebraic entity $((R|R);\ \cdot, \vee, (\ )';\ 0, 1, (0|0); \leq)$ is such that $\cdot, \vee$ are associative, idempotent, commutative operations compatible with $\leq$ being a legitimate meet-join lattice ordering over $(R|R)$, bounded below by the zero element wrt $\cdot, \vee$: 0, are bounded above by the unit element wrt $\cdot$, $\vee$: 1. In addition, $(R|R)$ with this structure has $\cdot$ and $\vee$ being mutually distributive and absorbing, as well as involutive operation $(\ )'$ (though, not in general orthocomplemented, thereby eliminating $(R|R)$ here from being a boolean algebra as R is) such that $(\cdot, \vee, (\ )')$ is a DeMorgan triple.

(iv) Furthermore, it can be shown directly that $(R|R)$ is always relatively pseudocomplemented and hence pseudocomplemented. Specifically, denoting the relative pseudocomplement of $(C|D)$ wrt $(A|B)$ as $(C|D) \rhd (A|B)$ and the pseudocomplement of $(C|D)$ as $(C|D)^* \overset{d}{=} (C|D) \rhd 0$, and recalling the well-known results (see Mendelson (1970, p. 182 et passim)) that relative to R, $B \rhd A = B \Rightarrow A$ and $B = B'$,

$$(C|D) \rhd (A|B) = \lambda \vee (A|B) = (\lambda \vee A|\lambda \vee B);\ \lambda \overset{d}{=} B'D' \vee C'D;\ (C|D)^* = C'D, \quad (6.86)$$

reducing to the corresponding unconditional situation for R, when $D = B = 1$. The pseudocomplement mapping $(\ )^* : (R|R) \to R$ satisfies the Stone identity

$$(A|B)^* \vee (A|B)^{**} = 1, \text{ all } (A|B) \in (R|R), \quad (6.87)$$

showing $((R|R);\ \cdot, \vee, 0, 1; \leq; (\ )^*)$ is a Stone algebra. Referring, e.g., to Grätzer

(1978), the *skeletal* and *dense sets of* $(R|R)$ are, respectively,

$$(R|R)^{*} \overset{d}{=} \{(A|B)^{*} : (A|B) \in (R|)\} = R;\ D(R|R) \overset{d}{=} (\ )^{*-1}(0) = \mathcal{U} \cup \{(0|0)\}, \qquad (6.88)$$

yielding readily the relations (see also (6.82))

$$D(R|R) = (R|R)^{*} \vee (0|0);\ (A|B)^{*\prime} = (A|B)^{**}, \text{ all } (A|B) \in (R|R);\ (0|0)^{\prime *} = 0. \qquad (6.89)$$

(v) Converse to (iii) and (iv) above, if $(R|R)$ is replaced by an abstract space S and similarly for operations $\cdot$, $\vee$, $(\ )'$, special elements 0, 1, $(0|0)$, partial (lattice) order $\leq$, and pseudocomplement operation $(\ )^{*}$, so that $(S; \cdot, \vee, (\ )'; 0, 1, (0|0); \leq; (\ )^{*})$ is any abstract Stone algebra with involutive operation $(\ )'$ making $(\cdot, \vee, (\ )')$ a DeMorgan triple, such that the formal relations hold in (6.89), then it follows that S with the above structure is isomorphic to $(R|R)$, with the same algebraic operations and relations, where now $R = S^{*}$ (guaranteed to be a boolean algebra). Independent of the above result initially, it can be shown that if $m : R \to \mathcal{P}(\Omega)$ is the standard injective Stone isomorphism, where Ω is some set dependent upon R, for any given boolean algebra, then $(m|m) : (R|R) \to \mathcal{P}(\Omega)|\mathcal{P}(\Omega))$ is also an isomorphism, extending m, relative to the conditional event algebra structure obtained via functional image extensions of the boolean operations for R and $\mathcal{P}(\Omega)$, where

$$(m|m)(A|B) \overset{d}{=} (m(A)|m(B)), \text{ all } (A|B) \in (R|R). \qquad (6.90)$$

Finally, if the above isomorphic representation of S by $(S^{*}|S^{*})$ is written $h : S \to (S^{*}|S^{*})$, then it follows that the composition $(m|m)\circ h : S \to (\mathcal{P}(\Omega)|\mathcal{P}(\Omega))$ is an injective isomorphism (Ω dependent on S^{*}), showing a full extension of the Stone Representation Theorem for all such *abstract conditional event algebras.*

(vi) *Higher order conditional events*, i.e., formal quantities $((A|B)|(C|D))$ can be given meaning and reduced, in effect, to single conditional events by use of the relative pseudoinverse operation, where $A, B, C, D \in R$ are arbitrary. This is based upon the following observation resulting from eq. (6.45) applied to the definition of conditional events:

$$(A|B) = \{x : x \in R \ \&\ xB = AB\} \qquad (6.91)$$

is the solution set of the conjunctive equation $xB = AB$, which has great intuitive appeal. Thus, it is perfectly reasonable to define the higher order conditional event

$$((A|B)|(C|D)) \overset{d}{=} \{(x|y) : (x|y) \in (R|R) \ \&\ (x|y)\cdot(C|D) = (A|B)\ (C|D)\}. \qquad (6.92)$$

But it follows from the theory of linear equations in relatively pseudocomplemented lattices (which $(R|R)$ is) (see Gratzer (1978) or Goodman, Nguyen, Walker (1991)), (6.92) becomes

$$((A|B)|(C|D)) = (R|R)\cdot((C|D) \triangleright ((A|B)\cdot(C|D)) \vee ((A|B)\cdot(C|D)). \quad (6.93)$$

Noting that the class union operation $U : \mathcal{P}(\mathcal{P}(R)) \to \mathcal{P}(R)$ is a homomorphism wrt all functionally-imaged extended operations over $\mathcal{P}(R)$ to those over $\mathcal{P}(\mathcal{P}(R))$, it follows that it is natural to inquire: What is the effect applying U to (6.93)? First, note that (6.86) with $(A|B)$ replaced by $(A|B)\cdot(C|D)$ can be shown to have the invariancy

$$(C|D) \triangleright ((A|B)|(C|D)) = \lambda_0 \vee ((A|B)\cdot(C|D)) = \lambda \vee (A|B) = (C|D) \triangleright (A|B),$$
(6.94)

but where now

$$\lambda_0 \overset{d}{=} (B \Rightarrow A)\cdot D' \vee C'D. \quad (6.95)$$

Thus, (6.93) becomes

$$\begin{aligned}((A|B)|(C|D)) &= (R|R)\cdot(\lambda_0 \vee ((A|B)\cdot(C|D))) \vee ((A|B)\cdot(C|D)) \\ &= (R|R)\cdot\lambda_0 \vee ((A|B)\cdot(C|D)), \end{aligned} \quad (6.96)$$

by distributivity and absorption properties of the operations.

Hence, applying U to (6.96), using its homomorphism properties and the calculus of operations from (6.58), (6.68), (6.96),

$$\begin{aligned}U((A|B)|(C|D)) &= U(R|R)\cdot\lambda_0 \vee (ABCD|r_2) = R\cdot\lambda_0 \vee (ABCD|r_2) \\ &= (0|\lambda_0') \vee (ABCD|r_2) = (ABCD|B\cdot(A'D' \vee CD)). \end{aligned} \quad (6.97)$$

Despite the nice algebraic properties of the above reduction, one drawback is that we do not have compatibility with probability in the sense

$$\begin{aligned}p((A|B)|(C|D)) &\overset{d}{=} p((A|B)\cdot(C|D))/p((C|D)) \\ &= p(ABCD|A'B \vee C'D \vee BD)/p(C|D) \\ &\neq p(U((A|B)|(C|D))), \text{ in general,}\end{aligned} \quad (6.98)$$

unlike the single conditional event case where no U is required. More work must be done in this area; forcing closure for higher order conditionals may lead to contradictions, analogous to Lewis' results (1976).

(vii) $(R|R)$ with the fundamental image extensions of operations on R can also be shown to be a modified version of Koopman's comparative conditional qualitative probability structure as discussed in Fine (1973, pp. 183-186).

(viii) Often, it is more appropriate to consider cartesian products or jointness of conditional events in place of direct conjunction, and similarly cartesian sums in place of disjunction. This is especially relevant when e.g. conditional events $(A_j|B_j)$, $j = 1,...,n$ are such that the B_j are all disjoint -- such as in flow chart instructions -- yet eqs. (6.74) and (6.75) show that the conjunction always lead to a trivial zero-type event, and hence zero probability evaluation, while disjunction always dually leads to the equally trivial unity case and a unit probability evaluation! Specifically, using the functional image extension approach as before, it can be shown that for any $(A_j|B_j) \in (R|R)$, $j = 1,..,n$,

$$(A_1|B_1) \times..\times (A_n|B_n) = (A_1 \times..\times A_n|B_1 \times..\times B_n) = (A_1B_1 \times..\times A_nB_n|B_1 \times..\times B_n), \tag{6.99}$$

$$(A_1|B_1) \dagger..\dagger (A_n|B_n) = ((A_1|B_1)' \times..\times (A_n|B_n)')' = ((A_1' \times..\times A_n')'|B_1 \times..\times B_n)$$
$$= (A_1 \dagger..\dagger A_n|B_1 \times..\times B_n). \tag{6.100}$$

Of course, with the use of cartesian products and sums, probability evaluations become more complex with joint probability specifications now required. Finally, note that no closure problems arise here, since all cartesian products -- and hence sums -- of boolean algebras are still boolean algebras.

(ix) The calculus of operations and relations obtained by functional image extensions of the boolean ones over R to $(R|R)$ also lead to a sound and complete *conditional probability logic of propositions* with the tautology class being $\mathcal{U}$ and the contradiction class being $\mathcal{Z}$. In connection with this, it can be shown that the only possible *boolean-like* function $f : (R|R)^2 \to (R|R)$, i.e.,

$$f((A|B), (C|D)) = (f_1(AB, B, CD, D)|f_2(AB, B, CD, D)), \text{ all } A, B, C, D \in R, \tag{6.101}$$

for some boolean functions $f_1, f_2 : R^4 \to R$ such that, in the spirit of (ii),

$$f((A|B), (C|D)) \in \mathcal{U} \text{ iff } (A|B) \leq (C|D); \text{ all } A, B, C, D \in R, \tag{6.102}$$

are $f = f^{(1)}$ and $f = f^{(2)}$, where for all $A, B, C, D \in R$,

$$f^{(1)}((A|B), (C|D)) \overset{d}{=} C'D \vee AB \vee B'D'; \quad f^{(2)}((A|B), (C|D)) \overset{d}{=} (C'D \vee AB|B \vee D). \tag{6.103}$$

f(1) is actually the consequent of the *natural isomorphic image* of Lukasiewicz three-valued logical implication, while $f^{(2)}$ is the natural isomorphic image of Sobocinski's three-valued logical material implication. (See Rescher (1969) for expositions on $Ł_3$ and Sob_3. The natural isomorphism connecting any three-valued logic and some choice of conditional event algebra is given below.)

(x) Other topics concerning conditional event algebras have begun to be developed, including: extension of random variables and relations with conditional random variables; problems of assignment of probability to conditional events relative to the functional image assignment of many values in light of the coset representation of conditional events as *sets* of events -- not the traditional single values (see also the latter part of sect. 7 here), extension of the idea of independence to conditional events (see also the Nguyen & Rogers paper in this monograph); and Fréchet-like bounds and probability expansions for various combinations of conditional events (in the same spirit as e.g. Hailperin (1984)).

As a final topic in this review, consider the natural isomorphism between any choice of conditional event algebra -- such as proposed here by functional image extensions, or that proposed commonly (but independently) by Schay, Adams, and Calabrese, or an alternative system also proposed by Schay, to be discussed briefly below -- and any corresponding choice of 3-valued (truth-functional) logic. [We will employ the abbreviation "ce-alg" for "conditional event algebra."]

Recall the operation construction technique of section 3, whereby a given bijection between two spaces X and Y with X having a given algebraic structure induces an isomorphism for Y now having the constructed algebraic structure. Of course, in general, one cannot guarantee that the constructed isomorphic operations over Y will be "recognizable" in some sense. Apropos to this, a basic connection was established between 3-valued indicator functions and conditional events as given in (6.62) -- basic bijection between $Mem_3(X)$ and $(R|R)$ -- and Theorem 6.3 (see also Remark (ii) following Theorem 6.4) -- characterization of ordering. However, no algebraic structure was imposed upon $Mem_3(X)$. Of course, since $Mem_3(X) \subseteq Mem(X)$ (up to the identification of (6.36)), the Zadeh-like operations and relations over the latter cna be used over the smaller class. (More on this later.) In response to the above remarks, the following theorem holds (Goodman (1990)) (see also Goodman et al,1991)):

Theorem 6.5: The three-valued indicator mapping as the natural isomorphism connecting all possible choices of conditional event algebras and all truth-functional three-valued logics.

First, denote the class of all n-ary boolean-like functions $f:(R|R)^n \to (R|R)$, analogous to the binary case given in (6.101), as $bool_n(R|R)$. Recall that the unit interval u is also used in effect as a single value between 0 and 1 and define

$$Q_o \overset{d}{=} \{0, u, 1\} \qquad (6.104)$$

as the common truth set of all three-valued logics to be considered. Any such logic is specified by some set of operations $\not{f}: Q_o^n \to Q_o \in Q_o^{Q_o^n}$. Then, there is a bijection $\theta: bool_n(R|R) \to Q_o^{Q_o^n}$ such that $1_.: ((R|R); bool_n(R|R)) \to (Q_o; Q_o^{Q_o^n})$ is an isomorphism: for all $(\underline{A}|\underline{B}) \overset{d}{=} ((A_1|B_1),..,(A_n|B_n)) \in (R|R)^n$,

$$1_{(\underline{A}|\underline{B})}(x) \overset{d}{=} (1_{(a_1|B_1)}(x),\ldots,1_{(A_n|B_n)}(x)),$$

for all $x \in X$, assuming $R \subseteq \mathcal{A}(X)$, and for all $f \in \mathrm{bool}_n(R|R)$,

$$1_{f(\underline{A}|\underline{B})}(x) = \theta f(1_{(A|B)}(x)). \quad (6.105)$$

Proof: The proof is completely constructive, enabling one to go from any three-valued logical operator to a corresponding conditional event one and vice-versa. (Again, see the cited references.) ∎

In connection with, and as an application of, the above theorem, consider briefly some of the approaches to defining conditional event operations extending the usual unconditional boolean ones, other than the functional image extension approach used so far -- and denoted from now on for convenience as the GN system. As mentioned before, independently Schay (1968), Adams (1975) and Calabrese (1987), denoted commonly as SAC, proposed identical ce-alg's. Actually, Schay also proposed an alternative ce-alg (same reference), which will be denoted simply as S. The complement, conjunction, and disjunction for these ce-alg's are, with the appropriate subscripting, for all $(A|B), (C|D) \in (R|R)$,

$$(A|B)'^{SAC} \overset{d}{=} (A|B)'^{S} \overset{d}{=} (A|B)'^{GN} = (A'|B). \quad (6.106)$$

$$(A|B) \vee_{SAC} (C|D) \overset{d}{=} (AB \vee CD|B \vee D);\ (A|B)\cdot_{SAC} (C|D) \overset{d}{=}$$
$$((A|B)' \vee_{SAC} (C|D)')', \quad (6.107)$$

a DeMorgan assumption, whence

$$(A|B)\cdot_{SAC}(C|D) = ((B \Rightarrow A)\cdot(D \Rightarrow C)|B \vee D) = (ABD' \vee B'CD \vee ABCD|B \vee D). \quad (6.108)$$

$$(A|B) \vee_S (C|D) = (AB \vee CD|BD);$$
$$(A|B)\cdot_S(C|D) = (A|B)' \vee_S (C|D)')' = (ABCD|BD). \quad (6.109)$$

Corollary 6.1. 3-valued logic characterizations of SAC, S, GN systems.

Under the mapping $1_{.}$, as in (6.105), the following isomorphisms hold between all operations defined for SAC, S, GN, and corresponding ones to be found in 3-valued logic:

$$\mathrm{SAC} \mapsto \mathrm{Sob}_3;\ \mathrm{S} \mapsto \mathrm{B}_3;\ \mathrm{GN} \mapsto Ł_3, \quad (6.110)$$

where Sob_3 is Sobocinski's 3-valued logic (see Rescher (1969, pp. 70, 342)), B_3 is Bochvar's 3-valued internal logic (Rescher (1969, pp. 29-34, 339)), and $Ł_3$ is Lukasiewicz' 3-valued logic (Rescher (1969, pp. 22-28, 335)).

Proof: Consequence of Theorem 6.5. ∎

Independently, Dubois & Prade (1989, 1990) have expressed interest in the development of conditional event algebra and, by informal means, pointed out the same correspondences as in Corollary 6.1, without recognizing the more general impact of Theorem 6.5. Recently (Goodman (1989)) a minisymposium was organized on conditional event algebras, as evidence also of the growing interest in the field.

Corollary 6.2. A characterization of GN.

Call any $f, g : (R|R)^2 \to (R|R)$ with f extending ordinary conjunction $\cdot$ over R and g extending ordinary disjunction $\vee$ over R, *monotone preserving* iff

$$1_{f((A|B),(C|D))} \leq 1_{(A|B)}, 1_{(C|D)};\ 1_{g((A|B),(C|D))} \geq 1_{(A|B)}, 1_{(C|D)} \tag{6.111}$$

pointwise over X (still assuming throughout here that $R \subseteq \mathscr{P}(X)$).

Also, for any operations $f, g : (R|R)^2 \to (R|R)$ extending $\cdot, \vee$ over R and $h : (R|R) \to (R|R)$ extending negation over R, call the system (f, g, h) a *common antecedent homomorphism* (or *coset compatible*) ce-alg iff for all $A, B, C \in R$,

$$h((A|B)) = (h(AB)|B);\ f((A|B),(C|B)) = (f(AB, CB)|B);$$

$$g((A|B),(C|B)) = (G(AB, CB)|B) \tag{6.112}$$

noting that necessarily $h(AB) = A'B$,

whence $$h((A|B)) = (A'|B)\ (=(A'B|B)). \tag{6.113}$$

Then:

(i) of the 81 possible binary boolean-like ce-alg's (f, g, h) extending ordinary conjunction, disjunction, negation, respectively, over R to $(R|R)$, which are DeMorgan for h such that

$$h((A|B)) = (A'|B), \text{ all } A, B \in R, \tag{6.114}$$

there are 4 which are also commutative and monotone preserving. Letting $f = \cdot_j$, $j = 1, 2, 3, 4$, for all $A, B, C, D \in R$,

$$(A|B)\cdot_1(C|D) = ABCD;\ (A|B)\cdot_2(C|D) = (ABCD|r_2 \vee B'D');$$

$$(A|B)\cdot_3(C|D) = (ABCD|B \vee D); \tag{6.115}$$

$$(A|B)\cdot_4(C|D) = (ABCD|r_2), \tag{6.116}$$

where r_2 is given in (6.73), noting that the SAC and S ce-alg's are not among this group, but GN is (determined through $\cdot_4$).

(ii) The unique boolean-like ce-alg extending ordinary conjunction, disjunction, and negation which is DeMorgan for extended negation h satisfying (6.114) and which is monotone preserving, possesses the common antecedent homomorphism property and for which its conjunction and disjunction extending operations are mutually distributive and idempotent is GN.

Proof: Consequence of Theorem 6.5. (Again, see the Goodman and Goodman, Nguyen, Walker references.) ∎

Remark. Since Lukasiewicz-$\aleph_1$ (min, max, 1 - ( )) logic is the core of Zadeh's fuzzy set operations relative to the space Mem(X), then Corollary 6.1 shows that the specialization of fuzzy sets and their Zadeh operations to $Ł_3$ for $M_3(X)$ is isomorphic to the GN conditional event algebra over $(R|R)$.

Flou and partitioning sets corresponding to conditional event indicator functions.

With the basic properties of conditional events and their operations and relations established and tied-in with conditional event indicator functions, it is of some interest to reinterpret eqs. (6.27)-(6.31), using the identifications of (6.36)-(6.38): For any A, B $\in$ R ($\subseteq$ P(X)), the corresponding flou set to $1_{(A|B)}$ is

$$\phi^{-1}(1_{(A|B)}) = \{\phi^{-1}(1_{(A|B)_t}) : t \in u\};\ \phi^{-1}(1_{(A|B)_t}) = ((\phi^{-1}(1_{(A|B)_t}))_s)_{s\in u}; \tag{6.117}$$

$$\text{for all } s \in u,\ (\phi^{-1}(1_{(A|B)_t}))_s = \begin{cases} X, & \text{if } s = 0, \\ B \Rightarrow A, & \text{if } 0 < s \le t, \\ A \cap B, & \text{if } t < s \le 1, \end{cases} \tag{6.118}$$

and the corresponding partitioning set is

$$(\phi\circ\psi)^{-1}(1_{(A|B)}) = \{(\phi\circ\psi)^{-1}(1_{(A|B)_t}) : t \in u\}, \tag{6.119}$$

with index set

$$J_{(\phi\circ\psi)^{-1}(1_{(A|B)_t})} = \{0, t, 1\}, \tag{6.120}$$

and for any $s \in \{0, u, 1\}$,

$$((\phi\circ\psi)^{-1}(1_{(A|B)_t}))_s = \begin{cases} B \dashv A, & \text{if } s = 0, \\ B', & \text{if } s = t, \\ A \cap B, & \text{if } s = 1. \end{cases} \tag{6.121}$$

Analogous to (6.38) for $\mathrm{Mem}_3(X)$, obtain through the identifications of (6.36)-(6.38),

$$\mathrm{Flou}_3(X) = \{\phi^{-1}(1_{(A|B)}) : (A|B) \in (R|R)\}, \tag{6.122}$$

$$\mathrm{Part}_3(X) = \{(\phi\circ\psi)^{-1}(1_{(A|B)}) : (A|B) \in (R|R)\}, \tag{6.123}$$

and clearly $\mathrm{Mem}_3(X)$, $\mathrm{Flou}_3(X)$, $\mathrm{Part}_3(X)$ are all bijective under the restrictions of ϕ, ψ, $\phi\circ\psi$ (analogous to the bijection part of Theorem 6.1 (ii)).

Again, from the remark following Corollary 6.2, the Zadeh fuzzy set operations for conjunction, disjunction, and negation, i.e., min, max, 1 - ( ), respectively, applied to

$Mem_3(X)$ component-wise yield an isomorphism with GN ce-alg: For $*$ = min, max, for all s, t $\in$ u,

$$(1_{(|B)_t} * 1_{(C|D)_t})_s = (1_{((A|B)\theta^{-1}(*)(C|D))_t})_s \;;\; 1 - (1_{(A|B)_t})_s = (1_{(A|B)'_t})_s, \quad (6.124)$$

i.e., component-wise over X

$$\min(1_{(A|B)}, 1_{(C|D)}) = 1_{(A|B)\cdot(C|D)};\ \max(1_{(A|B)}, 1_{(C|D)}) = 1_{(A|B)\vee(C|D)}; \quad (6.125)$$

$$1 - 1_{(A|B)} = 1_{(A|B)'} \quad (6.126)$$

with $(A|B)\cdot(C|D)$, $(A|B) \vee (C|D)$, and $(A|B)'$ all obtainable from GN as in equations (6.68) and (6.69).

Hence, the construction technique of section 3 yields the compatible results for the corresponding flou and partitioning sets, using (6.117)-(6.121) (first, flou):

$$\phi^{-1}(1_{(A|B)}) * \phi^{-1}(1_{(C|D)}) = \phi^{-1}(1_{(A|B)\theta^{-1}(*)(C|D)}), \quad (6.127)$$

i.e., for all s, t $\in$ u, $*$ = min, max, and noting (see sect. 4) $*$ = $\cap_{min}$, $\cup_{max}$, respectively,

$$(\phi^{-1}(1_{(A|B)_t}))_s * (\phi^{-1}(1_{(C|D)_t}))_s = (\phi^{-1}(1_{((A|B)\theta^{-1}(*)(C|D))_t}))_s$$

$$= \begin{cases} X, & \text{if } s = 0 \\ r_2 \Rightarrow (A\cap B\cap C\cap D) = (A\cap B\cap D') \cup (B'\cap C\cap D) \cup (B'\cap D') \cup & \\ \quad \text{if } 0 < s \le t, & \\ (A\cap B\cap C\cap D)\cap r_2 = A\cap B\cap C\cap D, & \\ \quad \text{if } t < s \le 1, & \end{cases} \quad (6.128)$$

for $*$ - min and $\varphi^{-1}(*) = \cdot$.;

$$= \begin{cases} X, & \text{if } s = 0 \\ q_2 \Rightarrow ((A\cap B) \cup (C\cap D)) = (A'\cap B\cap D') \cup (B'\cap C'\cap D) \cup (B'\cap D') & \\ \qquad \cup (A\cap B) \cup (C\cap D) & \\ \quad \text{if } 0 < s \le t, & \\ ((A\cap B) \cup (C\cap D)) \cap q_2 = (A\cap B) \cup (C\cap D) & \\ \quad \text{if } t < s \le 1, & \end{cases} \quad (6.129)$$

for $*$ = max and $\varphi^{-1}(*) = \vee$;

$$(\phi^{-1}(1_{(A|B)}))' = \phi^{-1}(1_{(A|B)'}), \quad (6.130)$$

i.e., for all s, t $\in$ u,

$$(\phi^{-1}(1_{(A|B)_t'}))_s = \begin{cases} X, & \text{if } s = 0 \\ B \Rightarrow A', & \text{if } 0 < s \le t, \\ B \dashv A, & \text{if } t < s \le 1. \end{cases} \tag{6.131}$$

Furthermore, by use of the basic identities

$$(B \Rightarrow A) \cap (D \Rightarrow C) = (r_2 \Rightarrow (A \cap B \cap C \cap D)); ((B \Rightarrow A) \cup (D \Rightarrow C)$$
$$= (q_2 \Rightarrow ((A \cap B) \cup (C \cap D)), \tag{6.132}$$

which, in their own right, are exact material implication parallels of the corresponding conditional event identities in (6.68) and (6.69), it follows by inspection of (6.118), that $*$ in (6.127)-(6.129), for $* = \min$, max, and correspondingly, for $\phi^{-1}(*) = \cdot, \vee$, relative to each of the three possible set values component-wise are isomorphic, i.e., symbolically,

$$\begin{pmatrix} X \\ B \Rightarrow A \\ A \cap B \end{pmatrix} * \begin{pmatrix} X \\ D \Rightarrow C \\ C \cap D \end{pmatrix} = \begin{pmatrix} X\, \theta^{-1}(*) X \\ (B \Rightarrow A)\ \theta^{-1}(*)\, (D \Rightarrow C) \\ (A \cap B)\ \theta^{-1}(*)\, (C \cap D) \end{pmatrix}. \tag{6.133}$$

Similarly, for the partitioning sets,

$$(\phi \circ \psi)^{-1}(1_{(A|B)}) * (\phi \circ \psi)^{-1}(1_{(C|D)}) = (\phi \circ \psi)^{-1}(1_{(A|B)\theta^{-1}(*)(C|D)}), \tag{6.134}$$

i.e., for all $t \in u$ and $s \in J_{(\phi \circ \psi)^{-1}(1_{((A|B)\theta^{-1}(*)(C|D))_t})} = \{0, t, 1\}$,

$$((\phi \circ \psi)^{-1}(1_{(A|B)_t}))_s * ((\phi \circ \psi)^{-1}(1_{)C|D)_t}))_s = ((\phi \circ \psi)^{-1}(1_{((A|B)\theta^{-1}(*)(C|D))_t}))_s$$

$$= \begin{cases} r_2 \dashv (A \cap B \cap C \cap D) = (A' \cap B) \cup (C' \cap D), & \text{if } s = 0, \\ r_2' = (A \cap B \cap D') \cup (B' \cap C \cap D) \cup (B' \cap D'), & \text{if } s = t, \\ (A \cap B \cap C \cap D) \cap r_2 = A \cap B \cap C \cap D, & \text{if } s = 1 \end{cases} \tag{6.135}$$

for $* = \min$ and $\phi^{-1}(*) = \cdot$;

$$= \begin{cases} q_2 \dashv ((A \cap B) \cup (C \cap D)) = A' \cap B \cap C' \cap D, & \text{if } s = 0, \\ q_2' = (A' \cap B \cap D') \cup (B' \cap C' \cap D) \cup (B' \cap D'), & \text{if } s = t, \\ ((A \cap B) \cup (C \cap D)) \cap (A \cap B) \cup (C \cap D), & \text{if } s = 1, \end{cases} \tag{6.136}$$

for $* = \max$ and $\phi^{-1}(**) = \vee$;

$$((\phi \circ \psi)^{-1}(1_{(A|B)}))' = (\phi \circ \psi)^{-1}(1_{(A|B)'}) \tag{6.137}$$

i.e., for all $t \in u$ and $s \in \{0, t, 1\}$

$$((\phi \circ \psi)^{-1}(1_{(A|B)'_t})) = \begin{cases} B \dashv A' = A \cap B, & \text{if } s = 0, \\ B' & \text{, if } s = t, \\ A' \cap B = B \dashv A, & \text{if } s = 1. \end{cases} \tag{6.138}$$

In summary:

Theorem 6.6. Apropos to eqs. (6.117)-(6.138), the following diagram holds, superseding Fig. 5.1 for the restriction to three values:

$$\begin{tikzcd} & (\mathrm{Flou}_3(X) ; \cdot, \vee, (\ )') \arrow[dr, "\phi"', "\text{isomorphism}"] & \\ (\mathrm{Part}_3(X) ; \cdot, \vee, (\ )') \arrow[ur, "\psi"', "\text{isomorphism}"] \arrow[dr, "\xi", "\text{isomorphism}"'] & & (\mathrm{Mem}_3(X); \ \min, \max, 1\text{-}(\ )) \\ & ((R|R); \ GN: \cdot, \vee, (\ )') \arrow[ur, "1_."', "\text{isomorphism}"] & \end{tikzcd}$$

$$\xi \overset{d}{=} 1_.^{-1} \circ \phi \circ \psi; \ R \subseteq \mathcal{P}(X) \text{ boolean algebra}; \ (\cdot, \vee) = (\cap_{\min}, \cup_{\max})$$

Figure 6.1. Summary of isomorphisms among $((R|R); GN)$, $\mathrm{Mem}_3(X)$, $\mathrm{Flou}_3(X)$, $\mathrm{Part}_3(X)$.

Proof: Combine the results of eqs. (6.117)-(6.138) with the compatible results of Theorem 6.1 (ii). ∎

7. Models and External Probabilities of Fuzzy Sets and Relations with Conditional Event Indicator Functions

The primary purpose of this section is to relate on a firm foundation the concept of a model as a consistent numerical evaluation relative to: fuzzy sets in general, and conditional events, in particular, and an appropriate fixed point relative · all membership functions at that point. In turn, this allows for a rationale to define probabilities for fuzzy sets, in general, and conditional events, in particular. To this end, assume throughout that boolean algebra $R \subseteq \mathcal{P}(X)$. Also, recall (Mendelson (1970)) the concepts of *filters* and *ultrafilters for* R. Call R *atomic*$_0$, if all finite subsets of R -- and hence all subsets of R whose complements are finite -- are in R, which immediately implies R is atomic. For such R, then essentially any ultrafilter $\mathcal{F}$ of R is in the principal ultrafilter form

$$\mathcal{F} \overset{d}{=} \mathcal{F}_x = \{A : x \in A \in R\} \subseteq R, x \in X. \tag{7.1}$$

In any case, denote the class of all ultrafilters of R as $\Omega(R)$. Also, define the mapping $\xi : \Omega(R) \times \mathrm{Flou}(X) \to u$, reminescent of the fundamental membership mapping ϕ (see (2.6)), where

$$\xi(\mathcal{F}, a) \overset{d}{=} \sup\{t : t \in u \ \&\ a_t \in \mathcal{F}\}, \text{ all } \mathcal{F} \in \Omega(R),\ a \in \mathrm{Flou}(X). \tag{7.2}$$

Next, define formally a *model of* Flou(X) as any function h: Flou(X) $\to$ u which is a homomorphism relative to $(\cap_{\min}, \cup_{\max}, (\ )')$ over Flou(X) and (min, max, $1 - (\ )$) over u, and hence over Mem(X). If, also h is an infinite homomorphism relative to $(\cup_{\max}, \max)$ and a homomorphism relative to $(\cap_{\mathrm{prod}}, \mathrm{prod})$ for the spaces $\mathrm{Flou}_1(X)$ and $\mathrm{Mem}_1(X)$, then call h a *strong model of* Flou(X). Finally, define

$$\mathrm{Mod}_0(X) \overset{d}{=} \{\xi(\mathcal{F}, \cdot) : \mathcal{F} \in \Omega(R)\} \tag{7.3}$$

and denote the class of all strong models of Flou(X) as $\tilde{\mathrm{Mod}}(X)$. Before giving the main theorem, the following lemma should be pointed out:

Lemma 7.1. If $R \subseteq \mathcal{P}(X)$ is atomic_0, then for any $\mathcal{F} \in \Omega(R)$, there is a unique corresponding $x \in X$ such that

$$\mathcal{F} = \mathcal{F}_x \ \&\ \xi(\mathcal{F}_x, a) = \phi(a)(x), \text{ all } a \in \mathrm{Flou}(X). \tag{7.4}$$

Proof: Use the definition of ϕ in (2.6), noting from (7.1), for $a = (a_t)_{t \in u}$,

$$a_t \in \mathcal{F}_x \text{ iff } x \in a_t, \text{ all } x \in X,\ t \in u. \tag{7.5}$$

■

Theorem 7.1 Basic characterization of strong models of Flou(X)

The equation
$$\tilde{\mathrm{Mod}}(X) = \mathrm{Mod}_0(X) \tag{7.6}$$
is true. More specifically, the following holds:

(i) For each $\mathcal{F} \in \Omega(R)$, $\xi(\mathcal{F}, \cdot)$. Flou(X) $\to$ u is a strong model of Flou(X). In particular, note the case when R is atomic_0, by Lemma 7.1, (7.4) holds.

(ii) For each $h \in \tilde{\mathrm{Mod}}(X)$,

$$\mathcal{F}_{(h)} \overset{d}{=} \{a : a \in R \ \&\ h(a) = 1\} = h^{-1}(1) \in \Omega(R) \tag{7.7}$$

and

$$\mathcal{h} = \xi(\mathcal{F}_{(h)}, \cdot) \in \mathrm{Mod}_0(X). \tag{7.8}$$

(iii) For each $\mathcal{h} \in \tilde{\mathrm{Mod}}(X)$, there is a unique $x_{\mathcal{h}} \in X$ such that

$$\mathcal{h}(a) = \phi(a)(x_{\mathcal{h}}), \text{ all } a \in \mathrm{Flou}(X). \tag{7.9}$$

If R is also atomic$_0$, then (7.8) and (7.9) combine to become

$$\mathcal{F}_{(\mathcal{h})} = \mathcal{h}^{-1}(1) = \mathcal{F}_{(\mathcal{h})_{x_{\mathcal{h}}}} = (\phi(\cdot)(x_{\mathcal{h}}))^{-1}(1);\ \mathcal{h}(a) = \xi(\mathcal{F}_{(\mathcal{h})_{x_{\mathcal{h}}}}, a) = \phi(a)(x_{\mathcal{h}}), \tag{7.10}$$

all $a \in \mathrm{Flou}(X)$.

(iv) $$\mathcal{h}(\phi^{-1}(c)) = c, \text{ all } c : X \to u \in \mathrm{Flou}_1(X). \tag{7.11}$$

Proof (i): Let $J \subseteq u$, $a^{(s)} \in \mathrm{Flou}(X)$, $f_s \overset{d}{=} \phi(a^{(s)})$, $s \in J$, and $\mathcal{F} \in \Omega(R)$, all arb. Define $\alpha \overset{d}{=} \xi(\mathcal{F}, \cup_{min}(a^{(s)}))$ and $\beta \overset{d}{=} \sup_{s\in J}(\xi(\mathcal{F}, a^{(s)}))$. Since $(\max_{s\in J}(f_s))^{-1}[t, 1] \geq f_s^{-1}[t, 1]$, all $s \in J$, then by the definition in (7.2), $\alpha \geq \beta$. Consider the converse: First, as a supremum, for all $\delta > 0$, there exists $s_\delta \in J$ with $f_{s_\delta}(x) \leq \sup_{s\in J}(f_s(x)) < f_{s_\delta}(x) + \delta$. Since $\mathcal{F}$ is a filter, if $A \overset{d}{=} \{x : x \in X \ \&\ \sup_{s\in J}(f_s(x)) \geq t\} \in \mathcal{F}$, then $B \overset{d}{=} \{x : x \in X \ \&\ f_{s_\delta}(x) \geq t - \delta\} \in \mathcal{F}$, since the above equation implies $A \subseteq B$. Then, for all $\delta > 0$, using (7.2), letting $C_{s,t} \overset{d}{=} \{x : x \in X * f_s(x) \geq t\}$,

$$\begin{aligned}\alpha &\leq \sup\{t : t \in u \ \&\ B \in \mathcal{F}\}) = \sup_{s\in J}\{\delta + \sup\{t - \delta \varepsilon[-\delta, 1 - \delta] \ \&\ C_{s,t-\delta} \in \mathcal{F}\} \\ &= \delta + \sup_{s\in J}\{\sup\{t : t \in [0, 1 - \delta] \ \&\ C_{s,t} \in \mathcal{F}\} \\ &= \delta + \sup_{s\in J}\{\sup\{t : t \in u \ \&\ C_{s,t} \in \mathcal{F}\} = \delta + \beta,\end{aligned}$$

implying that $\alpha \leq \beta$. Hence, $\alpha = \beta$ and thus $\xi(\mathcal{F}, \cdot)$ is an infinite homomorph. wrt $(\cup_{max}, \max)$ for spaces Flou(X) and Mein(X).

Next, for any $a^{(j)} \in \mathrm{Flou}(X)$ and $f_j \overset{d}{=} \phi(a^{(j)})$, $j = 1, 2$, arb., $\xi(\mathcal{F}, \min(f_1, f_2)) \geq \min(\xi(\mathcal{F}, f_1), \xi(\mathcal{F}, f_2))$, slightly abusing notation and using fact that $\mathcal{F}$ is a filter. Conversely, since $f_1^{-1}[t, 1] \cap f_2^{-1}[t, 1] \in \mathcal{F}$ implies $f_j^{-1}[t, 1] \in \mathcal{F}$, by ultrafilter property, the above inequality reverses, showing finally $\xi(\mathcal{F},\ )$ is a homomorphism wrt $(\cap_{min}, \min)$.

Next, letting $f \overset{d}{=} \phi(a)$, $t_0 \overset{d}{=} \xi(\mathcal{F}, a') = \sup\{t : t \in u \ \&\ D_t \in \mathcal{F}\}$,

$D_t \overset{d}{=} \{x \in X : f'(x) \geq t\}$, $t_1 \overset{d}{=} \xi(\mathscr{F}, a)' = \inf\{t : t \in u \ \&\ E_t \in \mathscr{F}\}$,

$E_t \overset{d}{=} \{x \in X : f'(x) \leq t\}$. By the definitions of sup and inf, for all $\delta > 0$, there are $t_{0\delta} \leq t_0 \leq t_{0\delta} + \delta$, $t_{1\delta} - \delta \leq t_1 \leq t_{1\delta}$ with $D_{t_{0\delta}} \in \mathscr{F}$, $D_{t_0+\delta} \notin \mathscr{F}$, $E_{t_1\delta} \in \mathscr{F}$, $E_{t_1-\delta} \notin \mathscr{F}$. This yields the intersection $\{x : x \in X \ \&\ t_{0\delta} \leq f'(x) \leq t_{1,\delta}\} \in \mathscr{F}$, from the filter property of $\mathscr{F}$; $t_{0\delta} \leq t_{1\delta}$. If $t_{1\delta} - t_{0\delta} > 2\delta$, one could pick midpoint $t_{2\delta}$, $t_{0\delta} < t_{2\delta} < t_{1\delta}$, with $t_{2\delta} - t_{0\delta} > \delta$ and $t_{1\delta} - t_{2\delta} > \delta$. Since $t_{2\delta} > t_{0\delta} + \delta > t_0$, $D_{t_{2\delta}} \notin \mathscr{F}$, by sup property of t_0. Since $\mathscr{F}$ is a filter, $D'_{t_{2\delta}} = \{x \in X : f'(x) < t_{2\delta}\} \subseteq E_{t_{2\delta}} \in \mathscr{F}$. Since $t_{2\delta} < t_{1\delta} - \delta < t_1$, then $E_{2\delta} \notin \mathscr{F}$, by inf property of t_1, a contradiction. Thus, $0 \leq t_{1\delta} - t_{0\delta} \leq 2\delta$, for all $\delta > 0$ arbitrary.
In turn, the above inequality implies, by the triangle inequality that

$$0 \leq t_1 - t_0 \leq t_1 - t_{1\delta} + t_{1\delta} - t_{0\delta} + t_{0\delta} - t_0 \leq \delta + 2\delta + \delta = 4\delta,$$

which by the arbitrariness of δ, finally implies that $t_1 = t_0$. Hence, $\xi(\mathscr{F}, \cdot)$ is a homomorphism wrt $((\ )', 1 - (\ ))$ for spaces Flou(X) and Mem(X). Finally, by the very definition of $\xi(\mathscr{F}, \cdot)$ applied to $\phi^{-1}(c)$, for any $c \in \mathrm{Mem}_1(X)$, $\xi(\mathscr{F}, \phi^{-1}(c)) = c$, completing the proof of (i).

Proof (ii): Eq. (7.7) follows from the basic properties of filters and ultrafilters. Next, consider the basic identity for any $f = \phi(a) \in \mathrm{Mem}(X)$:

$$f = \sup_{t \in u} \min(1_{f^{-1}[t,1]}, t) \quad (\text{over } X). \tag{7.12}$$

By properties of inverse functions,

$$a = \phi^{-1}(f) = \sup_{t \in u} \phi^{-1}(\min(1_{f^{-1}[t,1]}, t)),$$

whence for strong model $\mathscr{A}$,

$$\mathscr{A}(a) = \sup_{t \in u} \mathscr{A}(\phi^{-1}(\min(1_{f^{-1}[t,1]}, t))) = \xi(\mathscr{F}_{(\mathscr{A})}, a).$$

Proof (iii): Consider the identity

$$f = \sup_{x \in X}(f(x) \cdot \delta_x) \quad (\text{over } X), \tag{7.13}$$

where δ_x is the Krönecker delta function

$$\delta_x(y) = \begin{cases} 0, & \text{if } y = x, \ y \in X, \\ 1, & \text{if } y = x, \ y \in X. \end{cases} \tag{7.14}$$

Then, for $f \overset{d}{=} \phi(a)$, $a \in \mathrm{Flou}(X)$,

$$a = \phi^{-1}(f) = \sup_{x \in X} \phi^{-1}(f(x)) \cap_{\min} \phi^{-1}(\delta_x),$$

whence for strong model $\mathcal{h}$,

$$\mathcal{h}(a) = \sup_{x \in X} \min(\mathcal{h}(\phi^{-1}(f(x))), \mathcal{h}(\phi^{-1}(\delta_x)))) = \sup_{x \in X} \min(f(x), \mathcal{h}(\phi^{-1}(\delta_x))). \qquad (7.15)$$

The only remaining thing is to consider what values $\mathcal{h}(\phi^{-1}(\delta_x))$ can take:

<u>Case 1</u>. $\mathcal{h}(\phi^{-1}(\delta_x)) = 0$, all $x \in X$. But, this case implies by (7.15) that $\mathcal{h}(a) = 0$, for all $a \in \text{Flou}(X)$, contradicting the fact that for all $c \in u$, (7.11) holds.

<u>Case 2</u>. $\mathcal{h}(\phi^{-1}(\delta_x)) > 0$ for at least two distinct $x_j \in X$, $j = 1, 2$. But, since $\mathcal{h}$ is a homomorphism, wrt $\cap_{\min}$,

$$0 = \mathcal{h}(\phi^{-1}(0)) = \mathcal{h}(\phi^{-1}(\delta_{x_1} \cdot \delta_{x_2})) = \min(\mathcal{h}(\phi^{-1}(\delta_{x_1})), \mathcal{h}(\phi^{-1}(\delta_{x_2}))) > 0,$$

a contradiction!

<u>Case 3</u>. This is the only case left: there is a unique $x_{\mathcal{h}} \in X$ such that $\mathcal{h}(\phi^{-1}(\delta_{x_{\mathcal{h}}})) > 0$. Furthermore, since $\min(\delta_{x_{\mathcal{h}}}, \delta'_{x_{\mathcal{h}}}) = 0$, by homomorphism,

$$0 = \mathcal{h}(\phi^{-1}(0)) = \min(\mathcal{h}(\phi^{-1}(\delta_x)), \mathcal{h}(\phi^{-1}(\delta'_x))), \text{ implying } \mathcal{h}(\phi^{-1}(\delta'_{x_{\mathcal{h}}})) = 0,$$

whence, by the homomorphism property of $\mathcal{h}$ again,

$$\mathcal{h}(\phi^{-1}(\delta_{x_{\mathcal{h}}})) = 1. \qquad (7.16)$$

Finally, substituting (7.16) into (7.15) yields (7.9), provided that (7.11) is valid. The latter is simply so due to a variation of the standard Cauchy theorem (Aczél (1966, sect. 2.1 et passim)). ■

Corollary 7.1. Characterization of strong models when R is atomic$_o$.

Suppose that boolean algebra R ($\subseteq \mathcal{P}(X)$) is atomic$_o$. Then

$$\tilde{\text{Mod}}(X) = \{\phi(\cdot)(x) : x \in X\}, \qquad (7.17)$$

i.e., the strong models of Flou(X) coincide with the fundamental membership evaluations at each fixed point.

Proof: This is a restatement of the right side of (7.10) of Theorem 7.1.

■

The next results, specified to $M_2(X)$ and $M_3(X)$, can be developed without full use of the strong model assumption of Theorem 7.1. Throughout, suppose: boolean algebra $R \subseteq \mathcal{P}(X)$ is atomic$_o$; $\{0, 1\}$ is endowed with the usual classical logic (C_2) operations

$\cdot, \vee, (\,)' = 1 - (\,)$; $Q_0 = \{0, u, 1\}$ has the ordering $0 \le u \le 1$ and is given the $Ł_3$ or, equivalently, Zadeh) structure min, max, $(\,)' = 1 - (\,)$, where now

$$0' = 1,\ 1' = 0,\ u' = \{t' : t \in u\} = u. \tag{7.18}$$

Also, recall the equivalence of $\mathrm{Flou}_2(X)$, $\mathrm{Mem}_2(X)$, and $\mathcal{P}(X)$ by Theorem 6.1 (ii) (eqs. (6.17), (6.18)); one can restrict $\mathrm{Flou}_2(X)$ and $\mathrm{Mem}_2(X)$ suitably so that $\mathcal{P}(X)$ can be replaced in effect by R. Recall, also with R replacing $\mathcal{P}(X)$, the equivalence of $\mathrm{Flou}_3(X)$, $\mathrm{Mem}_3(X)$, and $((R|R); GN)$ (Theorem 6.6). Thus, the definition of a model remains well-defined when any of the equivalent spaces are interchangeably used. Note also the natural identification for any model $\mathscr{A}$

$$\mathscr{A}(\{x\}) = \mathscr{A}(\delta_x) = \mathscr{A}(x), \text{ for all } x \in X. \tag{7.19}$$

Let $\mathrm{Mod}_2(X)$ denote the class of all models of R (i.e., $\mathrm{Flou}_2(X)$), *excluding those models identically zero over all singletons of X -- and hence identically zero over all finite subsets of X, all in R.* Similarly, denote $\mathrm{Mod}_3(X)$ as the class of all models of $\mathrm{mem}_3(X)$ (i.e., $((R|R); GN)$, etc.) with the same type of exclusion as for $\mathrm{Mod}_2(X)$.

Theorem 7.2. Suppose all of the above hold. Then:

(i) $$\mathrm{Mod}_2(X) = \{1_{(\cdot)}(x) : 1_{(\cdot)} \text{ restricted to } R,\ x \in X\}. \tag{7.20}$$

Hence, for any $\mathscr{A}: R \to \{0, 1\}$ model of R, there is a unique $x_{\mathscr{A}} \in X$, such that

$$\mathscr{A}(A) = 1_A(x_{\mathscr{A}}), \text{ all } A \in R. \tag{7.21}$$

(ii) $$\mathrm{Mod}_3(X) = \{1_{(\cdot)}(x) : 1_{(\cdot)} \text{ restricted to } (R|R),\ x \in X\}. \tag{7.22}$$

Hence, for any $\mathscr{A}: (R|R) \to Q_0$ a model of $((R|R); GN)$, there is a unique $x_{\mathscr{A}} \in X$ such that

$$\mathscr{A}((A|B)) = 1_{(A|B)}(x_{\mathscr{A}}), \text{ all } (A|B) \in (R|R). \tag{7.23}$$

Proof (i): If there exists $x_1, x_2 \in X$ with $x_1 \ne x_2$ and $\mathscr{A}(x_1), \mathscr{A}(x_2) > 0$, then necessarily $\mathscr{A}(x_1) = \mathscr{A}(x_2) = 1$, implying $0 = \mathscr{A}(\emptyset) = \mathscr{A}(\{x_1\}) \cap \{x_2\}) = \min(\mathscr{A}(x_1), \mathscr{A}(x_2))$, implying $\mathscr{A}(x_1) = 0$ or $\mathscr{A}(x_2) = 0$, a contradiction. Thus, there is a unique $x_{\mathscr{A}}$ with $\mathscr{A}(x_{\mathscr{A}}) > 0$, i.e., $\mathscr{A}(x_{\mathscr{A}}) = 1$. Next, let $A \in R$ arbitrary. If $x_{\mathscr{A}} \in A$, then

$$\mathscr{A}(A) = \mathscr{A}(\{x_{\mathscr{A}}\} \cup A - \{x_{\mathscr{A}}\}) = \max(\mathscr{A}(x_{\mathscr{A}}), \mathscr{A}(A - \{x_{\mathscr{A}}\})) = \max(1, \mathscr{A}(A - \{x_{\mathscr{A}}\})) = 1. \tag{7.24}$$

If $x_{\mathscr{A}} \notin A$, then $x_{\mathscr{A}} \in A'$, whence by replacing A by A' in (7.24), one obtains

$$1 - \mathscr{A}(A') = 1 - \mathscr{A}(A), \text{ implying } \mathscr{A}(A) = 0. \tag{7.25}$$

Thus, (7.24) and (7.25) show (7.21).

Conversely, for any $x \in X$, it follows from standard properties that, in fact, $1_{(\cdot)}(x) : (R; \cdot, \vee, (\,)') \to (\{0, 1\}; C_2)$ is a homomorphism.

Proof (ii): First note from eq. (6.81) the identity here

$$(A|B) = (A \cap B) \cup (B' \cdot (\emptyset|\emptyset)), \text{ all } A, B \in R, \tag{7.26}$$

with corresponding indicator function form (see also (6.34)

$$1_{(A|B)}(x) = \max(1_{A \cap B}(X), \min(1_{B'}(x), u)), \text{ all } x \in X. \tag{7.27}$$

Now, it follows readily that the restriction of any model $\mathcal{A}$ of $((R|R); GN)$ to R is a model of R and hence part (i) above is valid. Thus there exists unique $x_{\mathcal{A}} \in X$ such that (7.21) holds. In addition, note that since $(\emptyset|\emptyset)$ has the (unique) property that $(\emptyset|\emptyset)' = (\emptyset|\emptyset)$, then applying $\mathcal{A}$,

$$\mathcal{A}((\emptyset|\emptyset)') = 1 - \mathcal{A}((\emptyset|\emptyset)) = \mathcal{A}((\emptyset|\emptyset)) \in Q_0, \tag{7.28}$$

implying by (7.18) that the only possible value of $\mathcal{A}((\emptyset|\emptyset))$ satisfying (7.28) is

$$\mathcal{A}((\emptyset|\emptyset)) = u. \tag{7.29}$$

Substituting, from the above reasoning, (7.21) and (7.29) into the evaluation of any $(A|B)$ by $\mathcal{A}$, using (7.26) and (7.27) yields

$$\begin{aligned} \mathcal{A}((A|B)) &= \max(\mathcal{A}(A \cap B), \min(\mathcal{A}(B'), \mathcal{A}((\emptyset|\emptyset)))) \\ &= \max(1_{A \cap B}(x_{\mathcal{A}}), \min(1_{B'}(x_{\mathcal{A}}), u)) = 1_{(A|B)}(x_{\mathcal{A}}). \end{aligned} \tag{7.30}$$

Eq. (7.30) shows (7.23) holding.

Conversely, Theorem 6.6 shows (or it can be shown directly) that for any $x \in X$, $1\ (x) : ((R|R); GN) \to (Q_0; \min, \max, (\ )')$ is a homomorphism, i.e., for all $A, B, C, D \in R$,

$$1_{(A|B)\cdot(C|D)}(x) = \min(1_{(A|B)}(x), 1_{(C|D)}(x)); \ 1_{(A|B)'}(x) = 1 - 1_{(A|B)}(x);$$

$$1_{(A|B)\vee(C|D)}(x) = \max(1_{(A|B)}(x), 1_{(C|D)}(x)). \tag{7.31}$$

■

External probabilities of fuzzy sets in general, and conditional events and their indicator functions, in particular.

Remarks. With the stage set by the above results, we can now give a natural interpretation to the definition of the *external probability of a fuzzy set* (callled in the fuzzy set literature simply the probability of a fuzzy event -- see e.g., Dubois & Prade (1980, pp. 141 et passim)):

Let $(\Lambda, \mathcal{A}, p)$ be a probability space, (X, R) a measurable space, and $W : \Lambda \to X$ some corresponding random variable. Then, W is not only induces the ordinary probability space $(X, R, p_\circ W^{-1})$, but more generally the space $(X, \mathrm{Flou}(X), p_\circ W^{-1})$, where for any $a \in \mathrm{Flou}(X)$,

$$(p_o W^{-1})(a) \overset{d}{=} E_W(\phi(a)(W)) = \int_{\omega\in\Lambda} \phi(a)(W(\omega))dP(\omega) = \int_{x\in X} \phi(a)(x)dP(W^{-1}(x)), \tag{7.32}$$

can now be interpreted as the $p_o W^{-1}$ -- *averaged model value of flou set* a (or equivalently, of fuzzy set membership function $\phi(a)$). Similarly, the "mean" of a,

$$"E"(a) \overset{d}{=} E_W(W \cdot \phi(a)(W)), \tag{7.33}$$

when $X \subseteq \mathbb{R}^n$ can be interpreted as the $p_o W^{-1}$ -- *averaged model-moment value of flou set* a, etc.

Finally, particularizing the above to conditional event indicator functions in view of the previous results connecting them to fuzzy set membership functions and flou sets, for any choice of A, B $\in R \subseteq \mathcal{P}(X)$, eq. (7.32) yields

$$(p_o W^{-1})(1_{(A|B)}) = E_W(1_{(A|B)}(W)) = 1 \cdot p(W^{-1}(A \cap B)) + 0 \cdot p(W^{-1}(B - A))$$
$$+ u \cdot p(W^{-1}(B'))$$
$$= p(W^{-1}(A \cap B)) + u \cdot p(W^{-1}(B')). \tag{7.34}$$

Interpreting u literally as the unit interval, makes (7.34) represent not just a single value but a range of values, so that denoting this further interpretation here by a hat, one obtains easily the closed interval

$$(p_o \hat{} W^{-1})(1_{(A|B)}) = [p(W^{-1}(A \cap B)), p(W^{-1}(A \cap B)), p(W^{-1}(B \Rightarrow A))], \tag{7.35}$$

using properties of inverse functions so that

$$p(W^{-1}(B \Rightarrow A)) = 1 - p(W^{-1}(B)) + p(W^{-1}(A \cap B)). \tag{7.36}$$

However, this leaves the basic problem of how to evaluate or replace this interval of values by a single one, which by inspection of the unconditional case should be p(A|B). If formally, u were assigned the value p(A|B) itself, it follows that substituting this ino (7.34) yields back p(A|B)! However, this formalism, as intuitively appealing it is, is still onbly a formal mechanism. A more satisfactory approach to this issue can be developed as follows (see also Goodman, Nguyen, Walker (1991) for a brief exposition):

A simple and natural way to assign a single figure-of-merit to a closed interval of real numbers is the computation of a weighted average of the upper and lower boundary points of the interval. Depending on the criterion chosen, the "optimal" choice of weights will vary. In general, the equally weighted mean need not be the choice -- unless a criterion such as the minimization with respect to that point in the interval of the sum of squared distances to every element of the interval is chosen. In line with developing an alternative criterion, consider first the following:

Theorem 7.3. In the following, let $\mathbb{R}$ denote the ordinary real line and let $s_o < t_o \in \mathbb{R}$ be arbitrary fixed and consider the closed interval $[s_o, t_o]$. Let I denote the class of all intervals $[s, t] \subseteq [s_o, t_o]$, with $s < t$. Define mapping $h_1 : \mathbb{R} \times I \to \mathbb{R}$, where

$$h_1(\lambda, [s, t]) \overset{d}{=} \lambda \cdot t + (1 - \lambda) \cdot s; \text{ all } \lambda \in \mathbb{R}, [s, t] \in I, \tag{7.37}$$

the *boundary-weighting function*, and for each positive integer $n \geq 2$, define recursively, the *n^{th} iterate of the boundary weighting function*

$$h_n(\lambda, [s, t]) \overset{d}{=} h_1(h_{n-1}(\lambda, [s, t]), [s, t]); \text{ all } \lambda \in \mathbb{R}, [s, t] \in I. \tag{7.38}$$

Also, define the special subclass of I,

$$I_o \overset{d}{=} \{[s, t] : [s, t] \in I \;\&\; t - s = 1\}. \tag{7.39}$$

Finally, consider for any $[s, t] \in I$, the *boundary-weighting invariance class*

$$H([s, t]) \overset{d}{=} \{\lambda : \lambda \in \mathbb{R} \;\&\; \text{eq. (7.41) holds for all } n\} \tag{7.40}$$

$$h_n(\lambda, [s, t]) = h_1(\lambda,. [s, t]); \; n = 1, 2, 3, . \tag{7.41}$$

With the above definitions established, it follows that:

(i) $$H([s, t]) = \emptyset, \text{ for all } [s, t] \in I_o \dashv \{u\}. \tag{7.42}$$

(ii) $$H([s, t]) = \mathbb{R}, \text{ for } [s, t] = u \tag{7.43}$$

(iii) $$H([s, t]) = \{\lambda_{s,t}\}, \text{ for all } [s, t] \in I \dashv I_o, \tag{7.44}$$

where

$$\lambda_{s,t} \overset{d}{=} s/(1 - t + s), \text{ for all } s, t \in \mathbb{R}, t - s \neq 0. \tag{7.45}$$

(iv) In cases (ii) and (iii), the fixed point property also holds

$$h_n(\lambda, [s, t]) = \lambda; \; n = 1, 2, 3, \ldots, \tag{7.46}$$

with λ arbitrary $\in \mathbb{R}$, for (ii) and $\lambda = \lambda_{s,t}$ (uniquely), for (iii).

(v) For all $[s, t] \in I - I_0$,

$$s \leq \lambda_{s,t} \leq t \quad \text{iff } [s, t] \subset u \quad \text{(proper inclus.)} \tag{7.47}$$

Proof: if $\lambda \in H(s, t)$, then, necessarily, choosing in (7.38) $n = 2$ and using (7.41) with $\lambda_1 \overset{d}{=} h_1(\lambda, [s, t])$,

$$\lambda_1 = h_1(\lambda_1, [s, t]) = \lambda_1 \; t + (1 - \lambda_1) \; s \tag{7.48}$$

Solving (7.48) for λ_1 immediately leads to the unique solution

$$\S_1 = \lambda_{s,t}, \text{ if } t - s \neq 1. \tag{7.49}$$

In turn, (7.49) through λ is

$$\lambda \cdot t + (1 - \lambda) \cdot s = \lambda_{s,t}, \tag{7.50}$$

which also yields the solution $\lambda = \lambda_{s,t}$, provided $t - s \neq 1$, showing (iii).

Returning back to (7.48), when $t - s = 1$, it becomes

$$\lambda_1 = \lambda_1 + s, \tag{7.51}$$

which, unless $s = 0$ -- whence $t = 1$ in this case and λ_1 can be arbitrary in $\mathbb{R}$, has no solution for λ_1. This shows (i) and (ii). (iv) follows for (iii) from (7.50), while for the case of (ii), it is obvious by inspection that (7.46) always holds. Finally, (v) is shown by consideration of the combination of possibilities $1 - t + s \gtreqless 0$ with $t \gtreqless 1$.

■

Note that Theorem 7.3 (iii) can be generalized in the following sense:

Theorem 7.4. Let $[s, t] \subseteq u$ with $s < t$ and $\lambda \in u$ arbitrary fixed, not necessarily in $H([s, t])$. Then,

$$h_\infty(\lambda, [s, t]) \overset{d}{=} \lim_{n \to \infty} h_n(\lambda, [s, t]) = \lambda_{s,t}, \tag{7.52}$$

with the sequence $(h_n(\lambda, [s, t]))_{n=1,2,\ldots}$ decreasing to, fixed at, increasing to $\lambda_{s,t}$, depending on whether $\lambda \geq \lambda_{s,t}$, $\lambda = \lambda_{s,t}$, $\lambda \leq \lambda_{s,t}$, respectively.

Proof: First, note that if $h_\infty(\lambda, [s, t])$ exists, then, taking limits as $n \to \infty$ in (7.38) yields

$$h_\infty = \lim_{n \to \infty} h_n = h_1(\lim_{n \to \infty} h_{n-1}, [s, t]) = h_1(h_\infty, [s, t]),$$

the same formally as in (7.48) with λ_1 replaced by h_∞. In summary,

$$h_\infty(\lambda, [s, t]) \text{ exists implies } h_\infty(\lambda, [s, t]) = \lambda_s \tag{7.53}$$

Next, analogous to (7.48) with equality replaced by inequality,

$$h_1(\lambda, [s, t]) \leq \lambda \quad \text{iff} \quad \lambda_{s,t} \leq \lambda. \tag{7.54}$$

In turn, (7.54) shows

$$h_2(\lambda, [s, t]) = h_1(h_1(\lambda, [s, t]), [s, t]) \leq h_1(\lambda, [s, t])$$

iff $\lambda_{s,t} \leq h_1(\lambda, [s, t])$ if $\lambda_{s,t} \leq \lambda$, solving for λ.

Continuing the above process shows the decreasing sequence

$$0 \leq \ldots \leq h_3(\lambda, [s, t]) \leq h_2(\lambda, [s, t]) \leq h_1(\lambda, [s, t]) \leq \lambda \text{ iff } \lambda_{s,t} \leq \lambda \tag{7.55}$$

(7.55) shows that $h_\infty(\lambda, [s, t])$ exists, if $\hat{\lambda}_{s,t} \leq \lambda$. The inequalities in (7.55) reverse, showing finally

$$h_\infty(\lambda, [s, t]) \text{ exists, for all } \lambda \in u. \tag{7.56}$$

Combining (7.53) and (7.56) shows (7.52). ∎

As a consequence of Theorems 7.3 and 7.4, call the assignment

$$h_o([s, t]) \overset{d}{=} h_1(\lambda_{s,t}, [s, t]) = \lambda_{s,t}, \text{ for } [s, t] \subset u, \tag{7.57}$$

the ***stable***, or ***fixed point, boundary-weighting average of*** $[s, t]$. Extending this idea further, if $\mathscr{A} \subset u$ is arbitrary, the ***stable boundary average of*** $\mathscr{A}$ is defined through the tightest closed interval around $\mathscr{A}$, $[\inf(\mathscr{A}), \sup(\mathscr{A})]$, provided that $\inf(\mathscr{A}) < \sup(\mathscr{A})$ and $[\inf(\mathscr{A}), \sup(\mathscr{A})] \subset u$:

$$h_o(\mathscr{A}) \overset{d}{=} h_o([\inf(\mathscr{A}), \sup(\mathscr{A})]). \tag{7.58}$$

As a basic application of the above, functionally extend a give prob. $p : R \to u$ ot $\hat{p} : \mathscr{P}(R) \to \mathscr{P}(u)$, analogous to the way ordinary boolean operations were extended from R to $\mathscr{P}(R)$ and then were restricted to the subclass $(R|R)$: (See again the discussion prior to Theorem 6.4)

$$\hat{p}(B) \overset{d}{=} \{p(A) : A \in B\}, \text{ for all } B \in \mathscr{P}(R). \tag{7.59}$$

Hence, (7.59) specializes to the following when $B = (A|B)$, for any $A, B \in R$, noting (6.45) and (6.51) show

$$(A|B) = \{y : y \in R \;\&\; A \cdot B \leq y \leq B \Rightarrow A\}, \tag{7.60}$$

whereby, using the monotonicity of unconditional probabilities,

$$\hat{p}((A|B)) = \{p(y) : y \in (A|B)\} \subseteq [p(A \cdot B), p(B \Rightarrow A)] \tag{7.61}$$

where

$$\inf(\hat{p}((A|B))) = p(A \cdot B) \;\&\; \sup(\hat{p}((A|B))) = p(B \Rightarrow A) = 1 - p(B) + p(A \cdot B). \tag{7.62}$$

All of this leads to

Theorem 7.5. Justification for assigning conditional probabilities to conditional events:
$p((a|b)) = p(a|b)$.

Let $p : R \to u$ be a given probability measure (R either a boolean algebra, or more strongly, a σ-algebra). Then, for all $A, B \in R$ such that $p(B) > 0$, the stable boundary average of $\hat{p}((A|B))$ coincides with $p(A|B)$.

Proof: In eqs. (7.57) and (7.58) let $\mathscr{A} = \hat{p}((A|B))$, $s = p(AB)$, $t = p(B \Rightarrow A)$, using (7.61) and (7.62), yielding

$$\begin{aligned} h_0(\hat{p}((A|B))) &= h_0([s, t]) = \lambda_{s,t} \\ &= p(AB)/(1 - p(B \Rightarrow A) + p(AB)) \\ &= p(AB)/p(B) = p(A|B), \end{aligned}$$

noting here that Theorem 7.3 (iii) is applicable, since $1 - p(B \Rightarrow A) + p(AB) = 0$ iff $p(B) = 0$, which does not hold here. ∎

Of course, other justifications for why $p(A|B)$ is interpreted as the ratio of antecedent to consequent probabilities, *from the standard viewpoint of conditional probabilities, not via conditional events*, are readily available such as the functional equation approach of Azcél (1966, pp. 319-324). See also the game-theoretic admissibility approach using conditional events, Lindsey (1982), Goodman et al (1991).

Returning to the computation of probabilities of conditional event indicator functions as part of the more general evaluation of probabilities of fuzzy sets, the basic quandary in eqs. (7.34) and (7.35) can now be solved reasonably. The difficulty with obtaining $(p_\circ W^{-1})(1_{(A|B)})$ is the presence of symbol or "third value" u, which if literally interpreted, yields the equally appearing difficult interval form $(p_\circ\hat{} W^{-1})(1_{(A|B)})$ in eq. (7.35). However, with the use of the stable boundary average of an interval, one now obtains easily

$$\begin{aligned} h_0((p_\circ\hat{} W^{-1})(1_{(A|B)})) &= h_0([p(W^{-1}(A \cap B)), p(W^{-1}(B \Rightarrow A))]) \\ &= p(W^{-1}(A \cap B))/(1 - p(W^{-1}(B \Rightarrow A)) + p(W^{-1}(A \cap B))) \\ &= p(W^{-1}(A \cap B))/p(W^{-1}(B)) \\ &= (p_\circ W^{-1})(A|B), \end{aligned} \tag{7.63}$$

using (7.36), a result that is naturally compatible with, and extends, the classical unconditional case

$$\begin{aligned} (p_\circ W^{-1})(1_A) &= E_W(1_A W)) = p(W^{-1}(A)) \\ &= (p_\circ W^{-1})(A); \ \text{all } A \in R. \end{aligned} \tag{7.64}$$

8. Summary of Random Set Representation of Fuzzy Sets.

The following development is a summary of results to be found in Goodman & Nguyen (1985, chpts. 5, 6). It is presented here only for purpose of ease of reference and as a background for the concept of conditional fuzzy sets given in the next section.

First, let $(\Lambda, \mathcal{A}, p)$ be a fixed probability space such that $\mathcal{U}: \Lambda \to \tilde{u}$ is a uniformly distributed random variable. Let $(X, \mathcal{B})$ be a fixed measurable space -- $\mathcal{B} \subseteq \mathcal{P}(X)$ is a σ-algebra, and hence a boolean algebra. For each $x \in X$, let $F_x(\mathcal{B}) \overset{d}{=} \{A : x \in A \in \mathcal{B}\}$ be the *filter class on* x *relative to* $\mathcal{B}$ and let $\tilde{\mathscr{C}} \subseteq \mathcal{P}(\mathcal{B})$ be any σ-algebra with $F_x(\mathcal{B}) \in \tilde{\mathscr{C}}$ for all $x \in X$. Call any mapping $S : \Lambda \to \mathcal{B}$ a *random subset of* X iff S is $(\mathcal{A}, \tilde{\mathscr{C}})$-measurable, in which case S induces the probability space $(\mathcal{B}, \tilde{\mathscr{C}}, p{\circ}S^{-1})$. Denote the class of all random subsets of X as $RS(X)$, distinguishing random subsets only if they differ in their probability evaluations. Denote the corresponding equivalence relation among random subsets of X as $\overset{dis}{=}$ for "equal in distribution". If $S \in RS(X)$ is such that range(S) $\in$ Flou(X), call S a *nested random subset of* X and denote the class of all such as $NRS(X)$ (up to equivalence $\overset{\cdot\cdot}{}$s). Also, identify $Mem(X)$ with the more restricted class of all functions in it which are actually $(\mathcal{B}, B_u)$-measurable. The following theorem is a conglomeration of results from Goodman & Nguyen (1985), modified for the definitions here:

Theorem 8.1. Summary of basic random set representations involving fuzzy sets.

Part 1.

(i) The *one point coverage function* $\nu : RS(X) \to Mem(X)$ is surjective, where

$$\nu(S)(x) \overset{d}{=} p(x \in S) \overset{d}{=} p(S \in F_x) = (p{\circ}S^{-1})(F_x), \quad \text{all } x \in X,\ S \in RS(X). \tag{8.1}$$

In particular, for any given $f \in Mem(X)$, one can choose (in general, non-unique)

$S = f^{-1}[\mathcal{U}, 1] \overset{d}{=} \{x : x \in X \ \&\ f(x) \geq \mathcal{U}$, i.e., for any $\omega \in \Lambda$,

$$S(\omega) = f^{-1}[\mathcal{U}(\omega), 1] = \{\omega_0 : \omega_0 \in \Lambda \ \&\ f(\omega_1) \geq \mathcal{U}(\omega)\}. \tag{8.2}$$

(ii) The following statements are equivalent:

(I) $S \in NRS(X)$

(II) There exists $f \in Mem(X)$ such that $S \overset{dis}{=} f^{-1}[\mathcal{U}, 1]$.

(III) There exists $a = (a_t)_{t\in u} \in \mathrm{Flou}(X)$ such that $S \overset{\mathrm{dis}}{=} a_{\mathcal{U}}$, where

$$a_{\mathcal{U}}(\omega) \overset{d}{=} a_{\mathcal{U}(\omega)}, \text{ for all } \omega \in \Lambda. \tag{8.3}$$

(IV) There exists $q = (q_t)_{t\in J_q} \in \mathrm{Part}(X)$ such that

$$S \overset{\mathrm{dis}}{=} q_{(\mathcal{U})} \overset{d}{=} \bigcup_{\substack{t\in J_q \\ \mathcal{U}\le t<1}} q_t. \tag{8.4}$$

(iii) For any choice of $f \in \mathrm{Mem}(X)$,

$$S \overset{d}{=} f^{-1}[\mathcal{U}, 1] = q_{(\mathcal{U})} \in \mathrm{NRS}(X), \tag{8.5}$$

where

$$q \overset{d}{=} (f^{-1}(t))_{t\in \mathrm{range}(f)} \in \mathrm{Part}(X), \tag{8.6}$$

noting that (I) and (II) are related via $f = \phi(a)$ (Theorem 2.1).

■

Motivated by the above results, denote for any $a \in \mathrm{Flou}(X)$ and any $q \in \mathrm{Part}(X)$, $a_{\mathcal{U}}$ as a *uniformly randomized flou set* and $f^{-1}(\mathcal{U})$, where $f \overset{d}{=} \phi(\psi(q))$, as a *uniformly randomized partitioning set*; denote the space of all uniformly randomized partitioning sets of X as $\mathrm{Part}(X; \mathcal{U})$. Also denote the obvious bijections where $a \to a_u$ and $q \to (\phi(\psi(q)))^{-1}(\mathcal{U})$, by the common notation $\mathrm{id}_{\mathcal{U}}: \mathrm{Flou}(X) \to \mathrm{Flou}(X; \mathcal{U})$ and $\mathrm{id}_{\mathcal{U}}: \mathrm{Part}(X) \to \mathrm{Part}(X; \mathcal{U})$.

Theorem 8.2. Summary of basic random set representations involving fuzzy sets: Part 2

The following diagram of isomorphisms holds, extending the isomorphisms of Figure 5.1 to the randomized spaces by use of $\mathrm{id}_{\mathcal{U}}$:

$$\begin{tikzcd}
\mathrm{Part}(X) \arrow[r, "\psi"] \arrow[d, "\mathrm{id}_{\mathcal{U}}"'] & \mathrm{Flou}(X) \arrow[r, "\phi"] \arrow[d, "\mathrm{id}_{\mathcal{U}}"'] & \mathrm{Mem}(X) \\
\mathrm{Part}(X; \mathcal{U}) \arrow[r, "\psi"] & \mathrm{Flou}(X; \mathcal{U}) \overset{\mathrm{dis}}{=} \mathrm{NRS}(X) \arrow[ur, "\nu"] &
\end{tikzcd}$$

Figure 8.1. Summary of isomorphisms for Part(X), Flou(X),

and their randomizatons and Mem(X). ∎

Relative to Fig. 8.1, the following relations hold for all $f \in \text{Mem}(X)$, $a \in \text{Flou}(X)$, $x \in X$:

$$\nu^{-1}(f) = \text{id}_{\mathcal{U}}(\phi^{-1}(f)) = (\phi^{-1}(f))_{\mathcal{U}} = ((f^{-1}[t, 1])_{t \in u})_{\mathcal{U}} = f^{-1}[\mathcal{U}, 1], \tag{8.7}$$

$$a_{\mathcal{U}} = \text{id}_{\mathcal{U}}(a) = \nu^{-1}(\phi(a)) = \phi(a)^{-1}[\mathcal{U}, 1], \tag{8.8}$$

$$f(x) = (\nu(\text{id}_{\mathcal{U}}(\phi^{-1}(f))))(x) = p(x \in (\phi^{-1}(f))_{\mathcal{U}}) = \nu(\nu^{-1}(f))(x) = p(x \in f^{-1}[\mathcal{U}, 1]), \tag{8.9}$$

$$\phi(a)(x) = \nu(\text{id}_{\mathcal{U}}(a))(x) = \nu(a_{\mathcal{U}})(x) = p(x \in a_{\mathcal{U}}) = \nu(\nu^{-1}(\phi(a)))(x) = p(x \in f^{-1}[\mathcal{U}, 1]). \tag{8.10}$$

Also, directly from sect. 4 replacing index variable t by r.v. $\mathcal{U}$, i.e., applying $\text{id}_{\mathcal{U}}$ (see eqs. (4.1), (4.5), (4.10))

$$a_{\mathcal{U}} \cap_{\text{cop}} b_{\mathcal{U}} = (\phi^{-1}(\text{cop}_o(\phi(a), \phi(b))))_{\mathcal{U}} = (a \cap_{\text{cop}} b)_{\mathcal{U}};\ a_{\mathcal{U}} \cup_{\text{cocop}} b_{\mathcal{U}} = (a \cup_{\text{cocop}} b)_{\mathcal{U}}, \tag{8.11}$$

similarly;

$$\nu(a_{\mathcal{U}} \cap_{\text{cop}} b_{\mathcal{U}}) = \phi(\text{id}_{\mathcal{U}}^{-1}(a_{\mathcal{U}} \cap_{\text{cop}} b_{\mathcal{U}})) = \phi(a \cap_{\text{cop}} b) = \phi(\phi^{-1}(\text{cop}_o(\phi(a), \phi(b)))$$
$$= \text{cop}_o(\phi(a), \phi(b)) = \text{cop}_o(\nu(a_{\mathcal{U}}), \nu(b_{\mathcal{U}})), \tag{8.12}$$

and

$$\nu(a_{\mathcal{U}} \cap_{\text{cop}} b_{\mathcal{U}})(x) = p(x \in (a_{\mathcal{U}} \cap_{\text{cop}} b_{\mathcal{U}}) = \text{cop}(p(x \in a_{\mathcal{U}}), p(x \in b_{\mathcal{U}})) = \text{cop}(\phi(a)(x), \phi(b)(x)), \tag{8.13}$$

$$\nu(a_{\mathcal{U}} \cup_{\text{cocop}} b_{\mathcal{U}})(x) = p(x \in (a_{\mathcal{U}} \cup_{\text{cocop}} b_{\mathcal{U}})) = \text{cocop}(p(x \in a_{\mathcal{U}}), p(x \in b_{\mathcal{U}}))$$
$$= \text{cocop}(\phi(a)(x), \phi(b)(x)); \tag{8.14}$$

$$a'_{\mathcal{U}} = (\phi^{-1}(1 - \phi(a)))_{\mathcal{U}} = (a')_{\mathcal{U}}, \tag{8.15}$$

$$\nu(a'_{\mathcal{U}}) = \phi(\text{id}_{\mathcal{U}}^{-1}(a'_{\mathcal{U}})) = \phi((\text{id}_{\mathcal{U}}^{-1}(a')_{\mathcal{U}})) = \phi(a') = \phi(a)' = \nu(a_{\mathcal{U}})', \tag{8.16}$$

$$\nu(a'_{\mathcal{U}})(x) = p(x \in a'_{\mathcal{U}}) = 1 - p(x \in a_{\mathcal{U}}) = p(x \in a'_{\mathcal{U}}), \tag{8.16'}$$

noting

$$p(x \in a_{\mathcal{U}}) = p(x \in \phi(a)^{-1}[\mathcal{U}, 1]) = p(\mathcal{U} \le \phi(a)(x)) = \phi(a)(x), \tag{8.17}$$

$$p(x \in b_{\mathcal{U}}) = p(x \in \phi(b)^{-1}[\mathcal{U}, 1]) = p(\mathcal{U} \le \phi(b)(x)) = \phi(b)(x).$$

When cop = min and cocop = max, all of the above can be strengthened as a direct uniformly randomized version of the Negoita-Ralescu representation (1975), extended here in Theorem 4.1. For example,

$$
\begin{aligned}
a_{\mathcal{U}} \cap_{\min} b_{\mathcal{U}} = (a \cap_{\min} b)_{\mathcal{U}} = a_{\mathcal{U}} \cap b_{\mathcal{U}} = \phi(a)^{-1}[\mathcal{U}, 1] \cap \phi(b)^{-1}[\mathcal{U}, 1] \\
= (\min(\phi(a)(\cdot), \phi(b)\phi(\cdot)))^{-1}[\mathcal{U}, 1] \\
= (\phi(a \cap_{\min} b))^{-1}[\mathcal{U}, 1],
\end{aligned} \tag{8.18}
$$

etc.

In the next development, the single space X is replaced by the family of spaces X_j, $j \in J$ and the single uniform random variable $\mathcal{U}$ is replaced by the stochastic process $\tilde{\mathcal{U}} \overset{d}{=} (\mathcal{U}_j)_{j \in J}$, for some finite or infinite index set J, where $\tilde{\mathcal{U}}$ is determined in distribution by some J-copula cop (with corresponding DeMorgan cocopula cocop), where each $\mathcal{U}_j : \Lambda \to u$ is uniformly distributed. Use the abbreviation

$\mathrm{comb}(\times_{cop}, \dagger_{cocop}; \underline{a})$ to denote any typical logical combination of $\underline{a} \overset{d}{=} (a^{(j)})_{j \in k}$, applying operations $\times_{cop}$, $\dagger_{cocop}$ in a well-defined way to $a^{(j)} \in \mathrm{Flou}(X_j)$, $j \in K \subseteq J$, finite. Use also the multivariable notation $\underline{x} \overset{d}{=} (x_j)_{j \in K}$, $\phi(\underline{a}) \overset{d}{=} (\phi(a^{(j)})_{j \in K}$, $\underline{\mathcal{U}} \overset{d}{=} (\mathcal{U}_j)_{j \in K}$, and e.g., the expressions

$$\underline{X} \overset{d}{=} (X_j)_{j \in K};\ (\underline{x} \in \phi(a)^{-1}[\underline{\mathcal{U}}, 1]) \overset{d}{=} ((x_j \in \phi(a^{(j)})^{-1}[\mathcal{U}_j, 1]))_{j \in K},$$

etc.

Theorem 8.3. Isomorphic-like evaluations of arbitrary logical combinations of flou sets through membership functions.

With the above assumptions it follows that if the combination is purely a repetitive $\times_{cop}$ or $\dagger_{cocop}$, then the results below are valid with this restriction. However, for the general case, the following holds:

(i) For $(\mathrm{cop}, \mathrm{cocop}) \in \{(\min, \max), (\mathrm{pord}, \mathrm{probsum})\}$, then for all $\underline{a} \in \mathrm{Flou}(\underline{X})$ and all $\underline{x} \in \underline{X}$,

$$
\begin{aligned}
(\phi(\mathrm{comb}(\times_{cop}, \dagger_{cocop}; \underline{a})))(\underline{x}) = p(\mathrm{comb}(\&, \mathrm{or}; (\underline{x} \in \phi(\underline{a})^{-1}[\underline{\mathcal{U}}, 1]))) \\
= p(\mathrm{comb}(\&, \mathrm{or}; (\underline{\mathcal{U}} \le \phi(\underline{a})(\underline{x}))))
\end{aligned}
$$

$$=p(\underset{j\in I_0}{\mathrm{or}}\ \underset{i\in I_j}{\&}\ (\mathcal{U}_i \le \phi(a^{(i)})(x_i)))$$

$$= \sum_{\emptyset \neq G \subseteq I_0} (-1)^{\mathrm{cart}(G)+1} \cdot \mathrm{cop}((\mathcal{U}_i \le \phi(a^{(i)})(x_i))_{(i \in I_j,\, j \in G)}, \tag{8.19}$$

for some index sets I_j, $j \in I_0$ determined by the combination.

(ii) For (cop, cocop) = (min, max), not only does (i) above hold, but in addition,

$$(\phi(\mathrm{comb}(\times_{\mathrm{cop}}, \dagger_{\mathrm{cocop}}; \underline{a})))(\underline{x}) = \mathrm{comb}(\mathrm{cop}, \mathrm{cocop})(\phi(\underline{a})(\underline{x})). \tag{8.20}$$

■

Note for logical combinations involving negations of compounds of flou sets, reduce by DeMorgan properties the negations to equivalent combinations of $\times$ and $\dagger$ of the flou sets so that essentially one has the original $\mathrm{comb}(\times, \dagger, (\ )'; \underline{a})$ replaced by the equivalent $\mathrm{comb}_0(\times, \dagger; \underline{b})$, where some of the $b^{(j)} = a^{(j)\prime}$ and the remaining $b^{(j)} = a^{(j)}$, $j \in K$.

9. Conditional Fuzzy Sets.

In the past, a number of individuals have attempted to define conditional fuzzy sets. Zadeh (1978, pp. 14-20), based on an analogy "though not completely" with conditional probability, simply defined conditioning as a kind of specification, not at all reducing to conditional probabilities. In particular, if $f : X \times Y \to u$ and f_2 is the Y-marginal of f, i.e.,

$$f_2(y) = \max_{x\in X} f(x, y), \quad \text{all } y \in Y, \tag{9.1}$$

then Zadeh's conditional fuzzy set (or possibility) function of f given f_2 at y is $f(\cdot, y) : X \to u$, i.e., formally the same as f itself with y fixed. Nguyen (1978) also proposed a conditional fuzzy set form not analogous to conditional probability. Nguyen made an assumption that the conditional form should be the ratio of the joint membership function to a function -- to be specified by a suitable criterion which he developed -- of *both* X- and Y-marginals, again, unlike conditional probabilities. This resulted in the form

$$f(y\,|\,x) = f(x, y)\cdot \max(1, f_1(x)/f_2(y))), \quad x \in X, \quad y \in Y. \tag{9.2}$$

Kosko (1986) reconsidering fuzzy entropy, also proposed that fuzzy conditioning could be identified as a "relative subsethood", which for discrete X = y is a single number, not a function of arguments

$$(f_1|f_2) \overset{d}{=} 1 - ((\sum_{x \in X} (\max(0, f_1(x)) - f_2(x)))/\sum_{x \in X} f_1(x))$$

$$= (\sum_{x \in X} \min(f_1(x), f_2(x)))/\sum_{x \in X} f_2(x). \tag{9.3}$$

Hisdal (1978) proposed the definition, for $f : X \times Y \to \bar{u}$, f_1 X-marginal,

$$f(y|x) \overset{d}{=} \begin{cases} f(x, y), & \text{if } f_1(x) > f(x, y), \\ [f(x, y), 1], & \text{if } f_1(x) = f(x, y), \end{cases} \tag{9.4}$$

for all $x \in X$, $y \in Y$.

Ramer (1989), on the other hand acknowledging the work of Hisdal and Nguyen, decided that for any $A \subseteq X$ finite and $f : X \to u$, letting

$A = \{x_1,..,x_m\}$, $X = \{x_1,..,x_m, x_{m+1},.., x_n\}$; $0 \le f(x_1) \le f(x_2) \le .. \le f(x_m) \le .. \le f(x_n) \le 1$,

$$f(x_i|A) \overset{d}{=} \begin{cases} f(x_i), & i = 1, .., m-1 \\ 1, & i = m. \end{cases} \tag{9.5}$$

From this, Ramer obtains some natural relations satisfied by Hisdal's proposed definition. In addition, he discusses the limiting continuous case and justifies the approach through a minimal cross entropy criterion relative to all possible functions on A. Bouchon (1987) proposed for any two functions $f : X \to u$ and $g : Y \to u$ the two types of conditional forms at any $x \in X$, $y \in Y$

(i) $(f(x)|g(y))_t \overset{d}{=} \sup\{t : t \in u \;\&\; \not{t}(g(y), t) \le f(x)\}$; $\not{t}: u^2 \to u$ cont. t-norm, (9.6)

with special cases

$$(f(x)|g(y))_{min} = \begin{cases} 1, & \text{if } f(x) \ge g(y) \\ f(x), & \text{if } f(x) < g(y) \end{cases}; \quad (f(x)|g(y))_{prod} = \min((f(x)/g(y)), 1) \tag{9.7}$$

the left hand side of (9.7) being the well-known intuitionistic implication (Rescher (1969, pp. 44, 45 et passim)).

(ii) $(f(x)|g(y))_{N_h} \overset{d}{=} \max(N_h(g(y)), f(x))$; $N_h(t) \overset{d}{=} h^{-1}(h(0) - h(t))$ a negation, (9.8)

where $h : u \to \mathbb{R}^+$ is nonincreasing continuous with $h(0) \le +\infty$ and $h(1) = 0$.

Approach (ii) is clearly a generalization of the use of material implication when

$h = 1 - (\ )$. Bouchon, among other properties discussed, points out

$$\mathcal{A}((f(x)\,|\,g(y)), g(y)) = \min(f(x), g(y)); \quad \mathcal{A}((f(x)\,|\,g(y))_{N_h}, g(y)) = \mathcal{A}(f(x), g(y)), \tag{9.9}$$

analogous to the usual condition satisfied by conditional probabilities.

Yager (1983) also discussed various approaches to extending or modifying classical material implication for fuzzy sets. (See also Sembi and Mamdani (1979) for a survey and analysis relative to fuzzy decision-making.)

In all of the above approaches, no appeal was made to probability theory, except for the obvious formal similarities. In fact, Mattila (1986) has concluded that fuzzy material implication is not the appropriate counterpart of conditional probability, in keeping with the distinction emphasized in this paper and others relative to the development of conditional event algebra. (Again, see section 6, following eq. (6.38).) Goodman & Stein (1989) attempted a definition for fuzzy conditioning, based upon the fuzzy set analogue of the basic characterization of conditional events as the solution set of a boolean linear equation -- see eq. (6.91). That is, if $\mathcal{A}$ is a generalization of Zadeh's classical $(\min, \max, 1 - (\ ))$ system over $\mathrm{Mem}(X)$ (called there a semi-boolean algebra, being a complete bounded distributive DeMorgan lattice) with conjunction $*$ and order $\leq$, for any $f, g \in \mathcal{A}$, the conditional form $(f\,|\,g)$ is given by

$$(f\,|\,g) \overset{d}{=} \{h : h \in \mathcal{A} \ \&\ h*g = f*g\}. \tag{9.10}$$

This led to the form, for any $f, g \in \mathrm{Mem}(X)$, using Zadeh's operations $(\min, \max, 1 - (\ ))$, for $x \in X$,

$$(f\,|\,g)(x) = \begin{cases} f(x) & , \text{ if } f(x) < g(x), \\ [g(x), 1], & \text{ if } f(x) \geq g(x), \end{cases} \tag{9.11}$$

reminiscent of Hisdal's earlier independent proposal (see (9.4)). Operations among such conditional entities were defined by use of the functional image technique, as shown earlier here for boolean operations extended to conditional event form (see remarks prior to Theorem 6.4). Unfortunately, unlike the boolean counterpart, closure of operations did not hold, i.e., the functionally extended form for min over conditional forms as in (9.10) did not lead back to the same conditional structure in (9.10).

It will be seen, however, that the approach taken here to defining conditional fuzzy sets comes closest to Bouchon's approach (i) for min = prod (see r.h.s. (9.7)), among all the proposed definitions. However, even in this case there is difference, as will be seen below.

With all of the above background established and *keeping in mind the random set* *representations of fuzzy sets as summarized in section 8*, the following new approach to fuzzy conditioning is proposed:

Suppose the <u>same</u> setting as in section 8 holds with $(\Lambda, \mathcal{A}, p)$ a fixed probability space, $\tilde{\mathcal{U}} = (\mathcal{U}_j)_{j\in J}$ a stochastic process of uniformly distributed random variables $\mathcal{U}_j : \Lambda \to u$ governed by copula cop, a collection of corresponding spaces $(X_j)_{j\in J}$ with flou spaces $\mathrm{Flou}(X_j)$ and membership function spaces $\mathrm{Mem}(X_j)$, $j \in J$, etc. Consider then w.l.o.g. any $a^{(j)} \in \mathrm{Flou}(X_j)$ and r.v. $\mathcal{U}_j$, $j = 1, 2$. Thus, as in section 8, there are the natural correspondences

$$a^{(j)} \mapsto \phi(a^{(j)})^{-1}[\,\mathcal{U}_j, 1] \mapsto (\,\mathcal{U}_j \le \phi(a^{(j)})) \mapsto \mathcal{U}_j^{-1}[0, \phi(a^{(j)})] \in \mathcal{A}^{X_j}$$

$$(a^{(1)} \times_{cop} a^{(2)}) \mapsto \phi^{-1}(\phi(a^{(1)})(\cdot) \times_{cop} \phi(a^{(2)})(\cdot\cdot)) \mapsto (\,\mathcal{U}_1 \le \phi(a^{(1)}) \;\&\; (\,\mathcal{U}_2 \le \phi(a^{(2)})$$
$$\mapsto ((\,\mathcal{U}_1, \mathcal{U}_2) \le (\phi(a^{(1)}), \phi(a^{(2)})))$$
$$\mapsto (\,\mathcal{U}_1^{-1}[0, \phi(a^{(1)})]) \cap (\,\mathcal{U}_2^{-1}[0, \phi(a^{(2)})]) \in \mathcal{A}^{X_1 \times X_2}, \tag{9.12}$$

where the exponentiation of $\mathcal{A}$ refers to the actual relations

$$\mathcal{U}_j^{-1}[0, \phi(a^{(j)})] = (\,\mathcal{U}_j^{-1}[0, \phi(a^{(j)})(x_j)])_{x_j \in X_j}, \quad j = 1, 2, \tag{9.13}$$

$$\mathcal{U}_1^{-1}[0, \phi(a^{(1)})] \cap \mathcal{U}_2^{-1}[0, \phi(a^{(2)})]$$
$$= (\,\mathcal{U}_1^{-1}[0, \phi(a^{(1)})(x_1)] \cap (\,\mathcal{U}_2^{-1}[0, \phi(a^{(2)})(x_2)]))_{x_j \in X_j}, \quad j = 1, 2 \tag{9.14}$$

Thus, the marginal flou sets, or equivalently, marginal membership functions correspond via marginal r.v. $\mathcal{U}_j$ to elements in $\mathcal{A}^{X_j}$ and the joint flou sets, correspond via joint r.v. $(\,\mathcal{U}_1, \mathcal{U}_2)$ to elements in $\mathcal{A}^{X_1 \times X_2}$. Since everything actually depends only on the range of values $\mathcal{A}$ for any choice of x_j, for the most part, we omit the x_1, x_2 arguments, but it will be always understood that these values are present consistently, i.e., for any choice of $(x_1, x_2) \in X_1 \times X_2$, for consequent and same x_2 for antecedent: We have already developed successfully an approach which converts unconditional events in a boolean algebra to conditional ones and allows for feasible computations for naturally extended boolean operations and relations to these conditional forms: namely conditional event algebra, as detailed in sections 6 and 7. Hence, it is proposed that the *conditional flou set* $(a^{(1)} | a^{(2)})_{cop}$, where $f_j = \phi(a^{(j)})$,

$j = 1, 2$, as usual, are identified with the ordinary conditional set,

$$(\alpha|\beta) \overset{d}{=} ((\mathcal{U}_1^{-1}[0, \phi(a^{(1)})]) \cap (\mathcal{U}_2^{-1}[0, \phi(a^{(2)})])| \mathcal{U}_2^{-1}[0, \phi(a^{(2)})]) \in (\mathcal{A} | \mathcal{A}), \tag{9.15}$$

where $(\mathcal{A} | \mathcal{A})$ is the conditional event algebra (with choice of operations such as GN or SAC, etc.) formed from σ-algebra $\mathcal{A}$, in precisely the same way $(R|R)$ was formed from R. Note also that $(\alpha|\beta)$ has a well-defined indicator function $1_{(\alpha|\beta)} : \Lambda \to Q_0 \overset{d}{=} \{0, u, 1\}$, where, as in (6.33),

$$1_{(\alpha|\beta)}(\omega) = \begin{cases} 1, \text{ if } \omega \in \mathcal{U}_j^{-1}[0, \ \phi(a^{(2)})] \cap \mathcal{U}_2^{-1}[0, \phi(a^{(2)})] \\ 0, \text{ if } \omega \in \mathcal{U}_2^{-1}[0, \ \phi(a^{(2)})] \dashv \mathcal{U}_1^{-1}[0, \phi(a^{(1)})] \\ u, \text{ if } \omega \in \Lambda \ \dashv \mathcal{U}_2^{-1}[0, \phi(a^{(2)})] \ . \end{cases} \tag{9.16}$$

Next, consider the probability evaluation of $(\alpha|\beta)$ by p, based on the usual procedure (see the discussion in section 7 following eq. (7.31) and basic equation (6.51′))

$$p((\alpha|\beta) = p(\alpha|\beta) = p(\alpha \cap \beta)/p(\beta), \ p(\beta) > 0, \tag{9.17}$$

where

$$p(\beta) = p(\mathcal{U}_2^{-1}[0, \phi(a^{(2)})]) = \phi(a^{(2)}), \tag{9.18}$$

since $\mathcal{U}_2$ is uniformly distributed over u, and

$$p(\alpha \cap \beta) = p(\mathcal{U}_1^{-1}[0, \phi(a^{(1)})] \cap \mathcal{U}_2^{-1}[0, \phi(a^{(2)})])$$
$$= \text{cop}_\circ(\phi(a^{-1})(\cdot) \ , \ \phi(a^{(2)})(\cdot\cdot)), \tag{9.19}$$

by the very definition of cop. Hence, when $p(\beta) > 0$,

$$(\alpha|\beta) = \text{cop}_\circ(\phi(a^{(1)})(\cdot), \phi(a^{(2)})(\cdot\cdot))/\phi(a^{(2)})(\cdot\cdot). \tag{9.20}$$

But, since, $\phi(a^{(j)}) = f_j$ is the usual membership function corresponding to flous et $a^{(j)}$, it is clear that (9.20) can be naturally interpreted as the conditional membership function of $\phi(a^{(1)})$ given $\phi(a^{(2)})$, when the latter is not zero. Finally, define for cop fixed:

For any $f_j \in X_j$, $j = 1, 2$, the conditional membership function $(f_1|f_2)_{cop} : X_1 \times X_2 \to u$, where

$$(f_1|f_2)_{cop}(x_1, x_2) \overset{d}{=} \text{cop}(f_1(x_1), f(x_2))/f(x_2), \quad x_j \in X_j, \tag{9.21}$$

$j = 1, 2$, provide at x_2, $f(x_2) > 0$. In order to make this compatible with the three-valued ordinary conditional event indicator function, define

$$(f_1 | f_2)_{cop}(x_1, x_2) \stackrel{d}{=} u, \quad \text{when } f_2(x_2) = 0. \tag{9.22}$$

Combining (9.21) and (9.22) yields the compact form for all $x_j \in X_j$,

$$(f_1 | f_2)_{cop} = (cop(f_1(x_1), f(x_2))/f(x_2)) \cdot \delta_{(f_2(x_2)>0)} + \delta_{(f_2(x_2)=0)} \cdot u, \tag{9.23}$$

where, analogous to (6.36), one makes the natural identification

$$(f_1 | f_2)_{cop} = \{(f_1 | f_2)_{cop,t} : t \in u\}, \tag{9.24}$$

where analogous to (6.33), $(f_1 | f_2)_{cop,t}$ is formally the same as $(f_1 | f_2)_{cop}$ with u replaced by t, for each $t \in u$.

Similarly, if one starts out with flou sets instead of membership functions, one can define

$$\phi((a^{(1)} | a^{(2)})_{cop}) \stackrel{d}{=} (\phi(a^{(1)}) | \phi(a^{(2)})) \tag{9.25}$$

and procede with $f_j \stackrel{d}{=} \phi(a^{(j)})$, $j = 1, 2$, as in (9.21) and on.

Note also the special case where $f_1 = 1_A$, $A \subseteq X_1$, $f_2 = 1_B$, $B \subseteq X_2$,

$$(1_A | 1_B)_{cop} = \max(1_{A\times B}, 1_{B'} \cdot u) = 1_{(A\times B | X_1 \times B)} = 1_{(A\times X_2 | X_1\times B)} \stackrel{d}{=} 1_{(A|B)}, \tag{9.26}$$

the r.h.s. of (9.25) being for two arguments, one in X_1 and the other in X_2. IF $X_1 = X_2 = X$ and $A, B \subseteq X$ and the arguments are restricted to be the same. $x_1 = x_2 = x$, then (9.25) becomes as in (6.34)

$$(1_A | 1_B)_{cop} = 1_{(A|B)} \quad \text{(single argument form)}, \tag{9.27}$$

showing, so far, compatibility of form of fuzzy conditional membership functions with the specialized conditional event indicator functions.

Next, returning to the motivating definition in (9.15), for conditional membershi functions identified as conditional events $(\alpha | \beta)$ in $(\mathscr{A} | \mathscr{A})$, it follows that the natural definition of any operation among membership functions is given through the counterpart over $(\mathscr{A} | \mathscr{A})$:

Analogous to the setting leading to Theorem 8.3, assume (cop, cocop) = (min, max) and that now $\tilde{\mathscr{U}} \stackrel{d}{=} (\mathscr{U}_{j,i})_{j\in J, i=1,2}$ is a uniformly distributed stochastic process over u, with probability space $(\Lambda, \mathscr{A}, p)$ fixed as before, $\mathscr{U}_{j,i} : \Lambda \to u$ uniformly distributed over u with $\tilde{\mathscr{U}}$ jointly governed by min Also, assume $(X_{j,i})_{j\in J, i=1,2}$ given spaces with each $X_{j,i}$ corresponding to $\mathcal{U}_{j,i}$ and that $a^{(j)} \in \mathrm{Flou}(X_{j,1})$, $b^{(j)} \in \mathrm{Flou}(X_{j,2})$ arb., $j \in K$.

Select an arbitrary index set $K \subseteq J$ and consider any well-defined logical combination of $(a^{(j)}|b^{(j)})_{min}$ through $\times_{min}$ and $\dagger_{max}$. Expressing this in multivariable notation where e.g., $(\underline{a}|\underline{b}) \overset{d}{=} ((a^{(j)}|b^{(j)})_{min})_{j\in K}$;

$\underline{\mathcal{U}}_i \overset{d}{=} (S\mathcal{F}_{j,i})_{j\in K}$; $\phi(\underline{a}|\underline{b})_{min} = (\phi(a)|\phi(b))_{min} = (\phi(a^{(j)})|\phi(b^{(j)}))_{min})_{j\in K}$;

$\underline{x}_i \overset{d}{=} (x_{j,i})_{j\in K}$, $x_{j,i} \in X_{j,i}$, $j \in J$, $i = 1, 2$, etc.,

$$
\begin{aligned}
\phi(\mathrm{comb}(\times_{min}, \dagger_{max}; (\underline{a}|\underline{b})_{min}))(\underline{x}) &\overset{d}{=} \mathrm{comb}(\times_{min}, \dagger_{max}; (\phi(\underline{a})|\phi(\underline{b}))_{min}(\underline{x})) \\
&\overset{d}{=} p(\mathrm{comb}(\cdot, \vee; (\underline{\mathcal{U}}_1^{-1}[\underline{0}, \phi(\underline{a})(\underline{x}_1)]|\ \mathcal{U}_2^{-1}[\underline{0}, \phi(\underline{b})(\underline{x}_2)]))) \\
&= p(\mathrm{comb}(\&, \mathrm{or},; ((\underline{\mathcal{U}}_1 \le \phi(\underline{a})(\underline{x}_1))|(\underline{\mathcal{U}}_2 \le \phi(\underline{b})(\underline{x}_2))))), \\
&= p(\alpha_0|\beta_0) = (p(\alpha_0 \cap \beta_0)|p(\beta_0))_{min},
\end{aligned}
\tag{9.28}
$$

with the right hand side of (9.28) interpreted in functional form dependent upon argument $\underline{x}$ and where conditional event $(\alpha_0|\beta_0) \in (\mathcal{A}|\mathcal{A})$ is obtained via the calculus developed out of Theorem 6.4 and evaluated via (6.51′). In particular, consider the single argument case where $K = \{1, 2\}$, $X_{j,i} = X$, $j = 1, 2$, $i = 1, 2$, and for convenience, let $a = a_1$, $b = b_1$, $c = a_2$, $d = b_2$. Then for $*_0 = \cdot\min$, $\vee \max$, $x \in X$, and letting

$$
\begin{aligned}
&\alpha \overset{d}{=} \mathcal{U}_{1,1}^{-1}[0, \phi(a)(x)] \leftrightarrow (\mathcal{U}_{1,1} \le \phi(a)(x));\ \beta \overset{d}{=} \mathcal{U}_{1,2}^{-1}[0, \phi(b)(x)] \leftrightarrow (\mathcal{U}_{1,2} \le \phi(b)(x)), \\
&\gamma \overset{d}{=} \mathcal{U}_{2,1}^{-1}[0 \cdot \phi(c)(x)] \leftrightarrow (\mathcal{U}_{2,1} \le \phi(c)(x)),\ \delta \overset{d}{=} \mathcal{U}_{2,2}^{-1}[0, \phi(d)(x)] \leftrightarrow (\mathcal{U}_{2,2} \le \phi(d)(x)),
\end{aligned}
\tag{9.29}
$$

$$
\begin{aligned}
\phi((a|b)_{min} *_o (c|d)_{min}) &= (\phi(a)|\phi(b))_{min} *_o (\phi(c)|\phi(d))_{min} \\
&= p((\alpha|\beta)*(\gamma|\delta)) \quad (\phi = \cdot, \vee) \\
&= \begin{cases} p(\alpha\beta\gamma\delta|r_2) = p(\alpha\beta\gamma\delta)/p(r_2), & \text{if } *_o = \cdot_{min} \quad (* = \cdot) \\ p(\alpha\beta \vee \gamma\delta|q_2) = p(\alpha\beta \vee \gamma\delta|q_2), & \text{if } *_o = \vee_{max} \quad (* = \vee) \end{cases}
\end{aligned}
\tag{9.30}
$$

where

$$
r_2 \overset{d}{=} \alpha'\beta \vee \gamma'\delta \vee \alpha\beta\gamma\delta;\quad q_2 \overset{d}{=} \alpha\beta \vee \gamma\delta \vee \alpha'\beta\gamma'\delta, \tag{9.31}
$$

using (6.68), (6.69), (6.73), and evaluation (6.51′).

Simplifying (9.30) and (9.31) using elementary probability properties,

$$p(\alpha\beta\gamma\delta) = \min(\phi(a)(x), \phi(b)(x), \phi(c)(x), \phi(d)(x))$$

$$p(\alpha\beta \vee \gamma\delta) = \min(\phi(a)(x), \phi(b)(x)) + \min(\phi(c)(x), \phi(d)(x)) - p(\alpha\beta\gamma\delta), \tag{9.32}$$

with $p(\alpha\beta\gamma\delta)$ given in terms of cop and membership functions as in (9.31).

$$p(r_2) = p(\alpha'\beta \vee \gamma'\delta) + p(\alpha\beta\gamma\delta);\ p(q_2) = p(\alpha\beta \vee \gamma\delta) + p(\alpha'\beta\gamma'\delta), \tag{9.33}$$

$$p(\alpha'\beta \vee \gamma'\delta) = p(\alpha'\beta) + p(\gamma'\delta) - p(\alpha'\beta\gamma'\delta), \tag{9.34}$$

$$p(\alpha'\beta) = \phi(b)(x) - \min(\phi(a)(x), \phi(b)(x));\ p(\gamma'\delta) = \phi(d)(x) - \min(\phi(c)(x), \phi(d)(x)), \tag{9.35}$$

$$\begin{aligned} p(\alpha'\beta\gamma'\delta) &= p(\beta\delta) - p(\beta\gamma\delta) - p(\alpha\beta\delta) + p(\alpha\beta\gamma\delta) \\ &= \min(\phi(b)(x), \phi(d)(x)) - \min(\phi(b)(x), \phi(c)(x), \phi(d)(x)) \\ &\quad - \min(\phi(a)(x), \phi(b)(x), \phi(d)(x)) + p(\alpha\beta\gamma\delta). \end{aligned} \tag{9.36}$$

Even simpler is the negation evaluation:

$$\begin{aligned} \phi((a|b)_{min}{}')(x) &= (\phi(a)|\phi(b))(x)' = p((\alpha|\beta)') = p(\alpha'|\beta) = 1 - p(\alpha|\beta) \\ &= 1 - (\phi(a)|\phi(b))(x), \end{aligned} \tag{9.37}$$

where

$$p(\alpha|\beta) = p(\alpha\beta)/p(\beta) = (\phi(a)(x), \phi(b)(x))/\phi(b)(x) \tag{9.38}$$

if $\phi(b)(x) > 0$.

Thus, (9.30)-(9.38) show that all extended boolean operations over conditional membership functions are closed -- due to the conditional event algebra evaluations -- and feasible to compute: being only simple arithmetic functions of the copula at certain subsets of $\{\phi(a)(x), \phi(b)(x), \phi(c)(x), \phi(d)(x)\}$.

Also, as a check, when $\phi(a) = 1_A$, $\phi(b) = 1_B$, $\phi(c) = 1_C$, $\phi(d) = 1_D$, for any choice of A, B, C, D $\subseteq$ X, it is easy to prove, via (9.30), (9.31), and (9.37), that (9.27) shows

$$(1_A|1_B)_{cop} = 1_{(A|B)},\ (1_C|1_D)_{cop} = 1_{(C|D)} \tag{9.39}$$

and for $*_o = \cdot_{min}, \vee_{max}$, corresponding to $* = \cdot, \vee$,

$$\begin{aligned} (1_A|1_B)_{min} *_o (1_C|1_D)_{min} &= 1_{(A|B)} *_o 1_{(C|D)} = 1_{(A|B)*(C|D)} \\ (1_A|1_B)_{min}{}' &= 1_{(A|B)}{}' = 1_{(A|B)'}, \end{aligned} \tag{9.40}$$

where the R.H.S. of (9.40) are the usual (GN) conditional event operations from Theorem 6.4, given in the indicator function form.

Note that from its very definition, conditional membership functions always satisfy the relations

$$(f|1)_{cop} = f;\ \text{if cop is assoc.,}\ (f|g)_{cop} = (cop_o(f, g)|g)_{cop};\ (f|g)_{cop} \cdot g = cop_o(f, g), \tag{9.41}$$

for all $f \in \mathrm{Mem}(X)$, $g \in \mathrm{Mem}(Y)$, and copula cop arbitrary; the dot in the right hand side of (9.41) being ordinary arithmetic product. Finally, it can also be verified directly, using the definition in (9.23) that for f and g as above, together with the assumption now that cop is associative and commutative (such as is the case for cop = min or prod) and Z is any third space and $h \in \mathrm{Mem}(Z)$ is arbitrary such that

$$\sup\{h(z) : z \in Z\} = 1 : \tag{9.42}$$

$$(f|g)_{cop} = \sup\{(f|cop_o(g, h(z))_{cop} \cdot (h(z)|g)_{cop} : z \in Z\} . \tag{9.43}$$

The result in (9.43) can be useful as an alternative to Bayes' theorem, where a parameter of interest is described by f, observed data corresponding to g, and auxiliary information in the form of attributes described by h, so that $(f|cop_o(g, h(z))_{cop}$ can be interpreted as an inference rule, while $(h(z)|g)_{cop}$ can be thought of as a conditional error form. In practice, both the inference rule and error form may have to be obtained directly, rather than be built up from the antecedent- consequent form, since these individual functions may not be known. The identity in (9.43) corresponds to the well-known expansion

$$p(x|y) = \int_{z \in Z} p(x|y, z) \cdot p(z|y)dz. \tag{9.44}$$

Applications of earlier versions of (9.43) to problems of data fusion (and track association, in particular) can be found in Goodman (1986). Further analysis and discussion of the above results may also be found in Goodman, Nguyen, & Walker (1991, chpt. 8).

Finally, it is of some interest to be able to determine the probability of a conditional fuzzy set. This should extend the unconditional case given in (7.32), as well as the modified conditional event indicator situation as presented in (7.32)-(7.36). There the ambiguity caused by the presence of the u term leads to an interval of probabilities, which was resolved by use of the stable boundary weighting average technique ((7.57), (7.45)), and justified by Theorems 7.3, 7.4). Motivated by the above, suppose that $(\Lambda, \mathcal{A}, p)$ is a fixed probability space, X, Y are given spaces, $(X \times Y, R)$ a measurable space, $W : \Lambda \to X \times Y$ a random variable, and $f \in \mathrm{Mem}(X)$, $g \in \mathrm{Mem}(Y)$ arbitrary, with copula cop fixed Then,

$$(p_o W^{-1})((f|g)_{cop} \overset{d}{=} E_W((f|g)_{cop}(W)) = c_1 \, e_1 + c_2 \cdot e_2, \tag{9.45}$$

by standard probability expansion, where also using (9.23),

$$c_1 \overset{d}{=} E_W((f|g)_{cop}(W)|g(W) > 0) = E_W(cop(f(W), g(W))|f(W) > 0), \tag{9.46}$$

$$c_2 \overset{d}{=} E_W((f|g)_{cop}(W)|g(W)=0) = u;\;\; e_1 \overset{d}{=} p(g(W)>0);\;\; e_2 \overset{d}{=} p(g(W)=0). \tag{9.47}$$

Hence, analogous to (7.35), substituting (9.47) into (9.45),

$$(p_\circ W^{-1})((f|g)_{cop}) = c_1 \cdot e_1 + e_2 \cdot u = [c_1 \cdot e_1, c_1 \cdot e_1 + e_2], \tag{9.48}$$

whence the stable boundary average yields

$$h_o((p_\circ W^{-1})((f|g)_{cop})) = (c_1e_1)/(1-(c_1e_1+e_2)+c_1e_1) = c_1e_1/(1-e_2) = c_1, \tag{9.49}$$

which checks with all special cases (including c.e. indicators, etc.).

Acknowledgements.

The author wishes to express his gratitude for the joint support of this work by: Code 1133 (under Dr. R. Wachter), Office of Naval Research; the Naval Ocean Systems Center Program for Research and Technology (under Dr. J. Silva, Code 014, and with Deputy Director for Research, Dr. A. Gordon, Code 0141, and Deputy Director for Exploratory Development, Dr. K.J. Campbell). Acknowledgements are also given here to the ongoing support of the line management at the Naval Ocean Systems Center (Code 42, Head, J. A. Salzmann, Jr. and Code 421, Head, M.M. Mudurian). Finally, the author wishes to thank: his colleague, Prof. H.T. Nguyen, Mathematics Dept., New Mexico State University at Las Cruces, for many valuable discussions, suggestions, and criticisms, especially during his summer 1990 stay at NOSC as an ASEE Summer Faculty Visitor; his new research associate at NOSC, Dr. P.G. Calabrese (NRC Senior Research Associate), for many comments and penetrating questions.

References.

1. Aczél, J. (1966), *Lectures on Functional Equations and Their Applications*, Academic Press, New York.
2. Adams, E. (1975), *The Logic of Conditionals*, D. Reidel, Dordrecht, Neth.
3. Barr, M. (1989), Fuzzy sets and Toposes, *Proc. 3rd Inter. Fuzzy Sys. Assoc.*, Univ. of Wash., Seattle, (Aug.), 225-228.
4. Belluce, L.P. (1986), Semisimple algebras of infinite valued logic and bold fuzzy set theory, *Canad. J. Math. 38(6)*, 1356-1379.
5. Boole, G. (1854), *An Investigation of the Laws of Thought*, Walton & Maberly, London. Reprinted by Dover Public., New York, 1951. (See especially Chpt. 6 et passim for conditioning concepts.)
6. Bouchon, B. (1987), Fuzzy inferences and conditional possibility distributions, *Fuzzy Sets & Sys. 23*, 23-41.

7. Bruno, G. & Gilio, A. (1985), Confronto fra eventi condizionati di probabililiti nulla nell' inferenza statistica bayesiana (in Italian with English summary), *Rivista di Matemat. per le Scienze Econom. e Soc.* (*Milano*) *8*(2), 141-152.

8. Copeland, A.H. (1950), Implicative boolean algebra, *Math. Zeitschr. 53*(*3*), 285-290.

9. Copeland, A.H. (1956), Probabilities, observations, and predictions, *Proc. 3rd Berkeley Symp. on Math. Stat. & Prob.* (*Vol. II*), 1954-1955 (J. Neyman, ed.), Univ. of CA Press, Los Angeles, 41-47.

10. Calabrese, P.G. (1987), An algebraic synthesis of the foundations of logic and probability, *Info. Sci. 42*, 187-237.

11. DeFinetti, B. (1974), *Theory of Probability, Vols. 1* & *2*, John Wiley, New York (Vol. 1: pp. 139, 140; Vol. 2: pp. 266, 267, 322).

12. Dubois, D. & Prade, H. (1980), *Fuzzy Sets & Systems*, Academic Press, New York.

13. Dubois, D. & Prade, H. (1989), Measure-free conditioning, probability, and non-monotonic reasoning, *Proc. 11th Inter. Joint Conf. AI*, Detroit, Mich. (Aug.), 1110-1114.

14. Dubois, D. & Prade, H. (1990), The logical view of conditioning and its application to possibility and evidence theories, *Inter. J. Approx. Reason. 4*, 23-46.

15. Eytan, M. (1981), Fuzzy sets: a topos-logical point of view, *Fuzzy Sets & Sys. 5*, 47-67.

16. Fine, T. (1973), *Theories of Probability: An Examination of Foundations*, Academic Press, New York.

17. Gaines, b. (1978), Fuzzy and probability uncertainty logics, *Info. & Control 38*, 154-169.

18. Gentilhomme, Y. (1968), Les ensembles flous en linguistique, *Cahiers de Ling. Théor. et Appliq. 5*, 47-65.

19. Glas, M de (1984), Representation of Lukasiewicz many-valued algebras. The atomic case, *Fuzzy Sets & Sys. 14*, 175-185.

20. Goguen, J.A. (1974), Concept representation in natural and artificial languages: axioms, extensions and applications for fuzzy sets, *Inter. J. Man-Machine Stud. 6*, 531-561.

21. Goodman, I.R. (1981), Fuzzy sets as random level sets: implications and extensions of basic results, in *Applied Systems & Cybernetics, Vol. VI* (G. Lasker, ed.), Pergamon Press, New York, 2757-2766.

22. Goodman, I.R. (1986), *PACT: An Approach to Combining Linguistic-Based and Probabilistic Information for Correlation and Tracking*, NOSC Tech. Doc. 878 (March), Naval Ocean Systems Center, San Diego.

23. Goodman, I.R. (1987), A measure-free approach to conditioning, *Proc. 3rd AAAI Workshop Uncert. in AI*, Univ. of Wash., Seattle (July), 270-277.

24. Goodman, I.R. (1989), Chair of "Conditional event algebras and conditional probabili: computations", minisymposium presented at SIAM Annual Meeting, San Diego, July 21, 1989. Speakers included I.R. Goodman, P.G. Calabrese, H.T. Nguyen, and D.W. Stein. Abstracts of presentations in *Final Program Book of Abstracts of SIAM Meeting*, pp. 19, 20, 34.

25. Goodman, I.R. (1990), Three valued-logic and conditional event algebras, *First Inter. Symp. Uncert. Model & Analy.*, Univ. of Md., College Park, to appear.

26. Goodman, I.R. (1991), Evaluation of combinations of conditioned information: I, a history, accepted for publication in *Info. Sci.*

27. Goodman, I.R. & Nguyen, H.T. (1985), *Uncertainty Models for Knowledge-Based Systems*, North-Holland Press, Amsterdam.

28. Goodman, I.R. & Nguyen, H.T. (1988), Conditional objects and the modeling of uncertainties, in *Fuzzy Computing* (M.M. Gupta & T. Yamakawa, eds.), North-Holland, New York, 119-138.

29. Goodman, I.R. & Nguyen, H.T. (1991), Foundations for an algebraic theory of conditioning, accepted for publication in *Fuzzy Sets & Sys.*

30. Goodman, I.R., Nguyen, H.T., & Rogers, G.S. (1991), On the scoring approach to admissibility of uncertainty measures in expert systems, accepted for publication in *J. Math. Analy. & Applic.*

31. Goodman. I.R., Nguyen, H.T., & Walker, E.A., *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*, monograph accepted for publication by North-Holland.

32. Goodman, I.R. & Stein, D.W., (1989), Extension of the measure-free approach to conditioning of fuzzy sets and other logics, *Proc. 3rd Int. Fuzzy Sys. Assoc.*, Univ. of Wash., Seattle (Aug.), 361-364.

33. Grätzer, G. (1978), *General Lattice Theory*, Birkhäuser Verlag, Basel.

34. Hailperin, T. (1984), Probability logic, *Notre Dame J. Formal Logic 25(3)*, 198-212.

35. Hailperin, T. (1986), *Boole's Logic and Probability, 2nd Ed.*, North-Holland Press, New York.

36. Hisdal, E. (1978), Conditional possibilities, independence, and non-interaction, *Fuzzy Sets & Sys. 1*, 283-297.

37. Höhle, U. (1982), A mathematical theory of uncertainty, in *Fuzzy Set and Possibility Theory: Recent Developments* (R.R. Yager, ed.), Pergamon Press, New York, 344-355.

38. Kosko, B. (1986), Fuzzy entropy and conditioning, *Info. Sci. 40*, 165-174.

39. Lewis, D. (1976), Probabilities of conditionals and conditional probabilities, *The Philos. Rev. 85(3)*, 297-315.

40. Lindley, D.V. (1982), Scoring rules and the inevitability of probability, *Inter. Statis. Rev. 50*, 1-26.

41. Mattila, J.K. (1986), On some logical points of fuzzy conditional decision making, *Fuzzy Sets & Sys. 20*, 137-145.

42. Mazurkiewicz, S. (1956), *Podstawy Rachunku Prawdopodoiénstwa* (in Polish -- Foundations of the Calculus of Probability), Akademia Nauk, Warsaw, Tom 32 (J. Łoś, ed.)

43. Mendelson, E. (1970), *Boolean Algebra and Switching Circuits*, Schaum's Outline Series, McGraw-Hill, New York.

44. Negoita, C.V. & Ralescu, D.A. (1975), Representation theorems for fuzzy concepts, *Kybernetes (G. Br.)1*, 169-174.

45. Nguyen, H.T. (1978), On conditional possibility distributions, *Fuzzy Sets & Sys. 1*, 299-309.

46. Nguyen, H.T. & Rogers, G.S. (1991), Conditional operators in a logic of conditionals, in this monograph.

47. Pitts, A.M. (1982), Fuzzy sets do not form a topos, *Fuzzy Sets & Sys. 8*, 101-104.

48. Radecki, T. (1977), Level fuzzy sets, *J. Cybern. 7*, 189-198.

49. Ralescu, D.A. (1979), A survey of the representation of fuzzy concepts and its applications, in *Advances in Fuzzy Set Theory & Applications* (M.M. Gupta, R.K. Ragade, & R.R. Yager, eds.) North-Holland, New York, 77-91.

50. Ramer, A. (1989), Conditional possibility measures, *Cybern. & Sys. 20*, 233-247.

51. Rényi, A. (1970), *Foundations of Probability*, Holden-Day, San Francisco.

52. Rescher, N. (1969), *Many-Valued Logic*, McGraw-Hill Co., New York.

53. Schay, G. (1968), An algebra of conditional events, *J. Math. Analy. & Applic. 24*(2) (Nov.), 334-344.

54. Schweizer, B. & Sklar, A. (1983), *Probabilistic Metric Spaces*, North-Holland Press, New York.

55. Sembi, B.S. & Mamdani, E.H. (1979), On the nature of implication in fuzzy logic, *Proc. 9th Inter. Symp. Multi. Logic*, Bath, G. Br., 143-149.

56. Stout, L.N. (1984), Topoi and categories of fuzzy sets, *Fuzzy Sets & Sys. 12*, 169-184.

57. Yager, R.R. (1983), On the implicative operator in fuzzy logic, *Info. Sci. 31*, 141-164.

58. Zadeh, L.A., (1965), Fuzzy sets, *Info. & Control 8*, 338-353.

59. Zadeh, L.A., (1978), Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets & Sys. 1*, 3-28.